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DISCUSSIONS

APPLICATIONS FOR ADMISSION AND TRANSFER

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Discussion . . . . . Nov., 1928, Jan., April, May, Aug., 1929,	May, 1930	Uncertain
Report of the Committee of the Irrigation Division on A National Reclamation Policy.	Sept., 1928	
Discussion (Committee's closure). May, Aug., Sept., Oct., Nov., Dec., 1929, Feb., April, Sept., 1930,	Jan., 1931	Closed
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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

DESIGN OF A REINFORCED CONCRETE SKEW ARCH

BY BERNARD L. WEINER,[1] JUN. AM. SOC. C. E.

SYNOPSIS

This paper consists of:

Part I.—Analysis of the Stresses in a Skew Arch: A Restatement, Approximation, and Extension of the Original Theory by J. Charles Rathbun, M. Am. Soc. C. E.; with a Numerical Example.

Part II.—Theory of Proportioning Sections; with a Numerical Example.

Part III.—Short-Cut for Computing Crown Stresses in a Symmetrical Skew "Rigid Frame" with Hinged Ends; with a Numerical Example.

The text contains a practical description of a method of designing skew arches. Parts of it are built upon the theory first presented by Professor Rathbun, with special adaptations developed from experience in the drafting-room.

Complete derivations of formulas are given for symmetrical and unsymmetrical, and for fixed and two-hinged arches. A theory for proportioning the sections is developed which is believed to be entirely new. An approximate "short-cut" for designing Rigid Frames also forms a part of the paper. Numerical examples are given to demonstrate the theory.

The writer has assumed that his readers have a working knowledge of the design of a right arch by standard methods.

INTRODUCTION

Modern traffic conditions make the construction of a skew bridge the rule rather than the exception. The past practice of locating sharp bends in a highway on each side of a bridge in order to eliminate the skew, is no longer feasible because of the high speed of the present automobile traffic. The necessity of a scientific method of designing skew arches of the solid barrel type is thus apparent.

NOTE.—Written discussion on this paper will be closed in **May, 1931**, *Proceedings*.

[1] Asst. Engr. and Structural Designer, Westchester County Park Comm., Bronxville, N. Y.

Analysis of the Stresses in a Skew Arch.—Until recently, very little was known about the action of a skew arch. In 1924, J. Charles Rathbun, M. Am. Soc. C. E., published his theory for the analysis of skew frames.[2] This paper gave a method for computing the total moments, thrusts, and shears acting on any section of the skew frame parallel to the abutments. The theory, as presented, applied only to symmetrical skew frames fixed at the abutments. At the request of Arthur G. Hayden, M. Am. Soc. C. E., Designing Engineer of the Westchester County Park Commission, Professor Rathbun rewrote and extended his theory to include symmetrical structures hinged at the abutments. The latter also furnished a computation for the crown stresses of a hinged, rigid-frame structure for the crossing of the Hutchinson River Parkway under Old Mamaroneck Road, Westchester County, New York (span, 56 ft.; skew angle, 33°). These manuscripts were checked by or under the direction of Henry G. Babcock, M. Am. Soc. C. E., of the Design Division of the Commission. This work has never been published.

By actual experience in the drafting-room, it was found advisable to revise the form of the equations for the analysis of the skew arch, and to make certain approximations suggested by Professor Rathbun. Part I of this paper concerns itself with this problem and also completes the theory by extending it to include unsymmetrical skew frames, both fixed and hinged at the abutments. A numerical example for a two-hinged, symmetrical rigid frame is also given.

Theory of Proportioning Sections.—The theory developed by Professor Rathbun gave merely the total moments, thrusts, and shears acting on a section of the skew frame parallel to the abutments. It did not treat the problem of distribution except in a general way. Various suggestions were made as to the distribution of the stresses produced by these forces and couples, and as to the method of proportioning the sections to resist these stresses. Upon examination, these suggestions were found to be unsatisfactory and, in most cases, scientifically unsound. Several theories for proportioning the sections were compared numerically and found to give widely varying results as to the amounts and locations of the steel areas. Part II proposes a theory for stress distribution and proportioning of the sections, which has been developed by the writer, and, although not entirely exact because of certain assumptions (especially with respect to the torsion moment), it is believed to be logical.

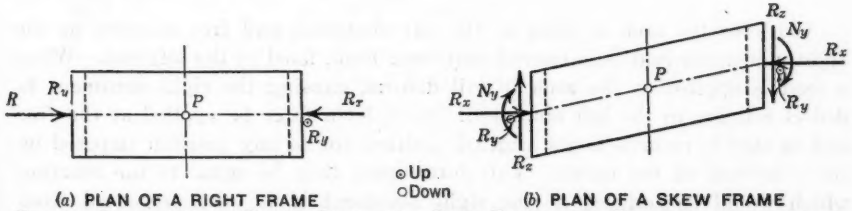
This part of the paper first develops the theory of the distribution of the stresses produced by the various forces and couples acting on a section of the skew frame. It then discusses the method of determining the location of the maximum stresses. In the case of a right arch the steel is perpendicular to the section to be designed which may be called the design plane. In the case of the skew arch, the steel may not be perpendicular to the section. The effect of the angle that the steel makes with the design plane upon the stresses and deformations is determined. The usual flexural formulas for designing a right reinforced concrete section, therefore, do not apply, and new formulas are developed. The application of these formulas is explained by numerical examples.

[2] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 611.

Crown Stresses in a Symmetrical Skew Rigid Frame with Hinged Ends.—The Westchester County Park Commission builds very extensively a type of structure known as the "Rigid Frame", developed by Mr. Hayden. After designing several symmetrical skew rigid frames with hinged ends, the writer found that certain approximations could be made because of the special shape of the frame. This approximation reduces the numerical work of computing the total moments, thrusts, and shears to not much more than that required for a right arch. This "short-cut" does not extend, however, to the proportioning of the sections. Since the "rigid frame" is now being used by others besides the Westchester County Park Commission, this "short-cut" is presented here with an illustrative numerical example. It is possible that a similar approximation can be made for the fixed symmetrical, and for the fixed and the hinged unsymmetrical, "rigid frame", but no data are available to determine the closeness of such approximations.

PART I.—ANALYSIS OF THE STRESSES IN A SKEW FRAME

The skew frame is a statically indeterminate structure containing forces and moments which in general cannot be made parallel to a single plane. They must be analyzed, therefore, in three dimensions instead of in two as is done in the case of the right frame. The skew of the structure introduces forces and moments which either do not exist at all in a right frame or are present only in a small degree. For example, compare the two structures shown in plan in Fig. (1).



(*a*) PLAN OF A RIGHT FRAME (*b*) PLAN OF A SKEW FRAME

FIG. 1.—COMPARISON OF FORCES ACTING IN A RIGID FRAME WITH THOSE IN A SKEW FRAME.

In Fig. 1(*a*) the vertical load, P, produces the horizontal reactions, R_x, which have the same line of action and, therefore, balance each other. In Fig. 1(*b*), the two reactions, R_x, do not tend to produce translation, but they form a couple which must be resisted by other reactions. This is done by the couple formed by the two horizontal shears, R_z, and by the twisting couples in both footings, N_y. Similarly, the vertical reactions, R_y, in Fig. 1(*b*), form a couple with the load, P, which is resisted by a couple at each footing in the vertical planes parallel to the abutments (not shown in the diagram). These couples in the footings produce unequal soil pressure between one end of the footing and the other, exactly similar to the case of a retaining wall. In the skew frame, therefore, there are three additional reaction components which do not exist in a right arch for a load placed symmetrically with respect to the spandrel walls. The numerical values of the reactions are statically indeterminate and depend upon the geometric and elastic properties of the

frame. These reactions produce three additional stresses throughout the arch which do not exist in a right arch for the condition of loading described.

In analyzing a right arch it is customary to take a section of unit width. In the case of a skew arch, the structure must be considered as a whole. In doing this, it is assumed that the load is distributed over the width of the arch ring; otherwise some error is involved. In the study prepared for the Westchester County Park Commission, Professor Rathbun states that recent experiments, some conducted by him and some by George E. Beggs, M. Am. Soc. C. E., show that in an arch the error in this assumption is negligible. As the frames investigated here are arched and not flat slabs, this assumption of distribution is allowable.

The theory presented herein differs in several minor respects from the original theory. In the analysis of a right arch, the shear and thrust deformations are usually neglected, because they are small compared with the bending deformation. Similarly, in the skew arch, two shear deformations, one thrust deformation, and the bending deformation about the radial axis at a point midway between the spandrel walls are neglected. The theory is therefore approximate, but it is sufficiently accurate for design purposes.

The system of signs used is different from that used in the original. This change is made after a thorough study of the question with the view of facilitating the use of the theory in the drafting-room. For a similar reason, some of the details of the derivations are also changed.

Method of Deriving Formulas

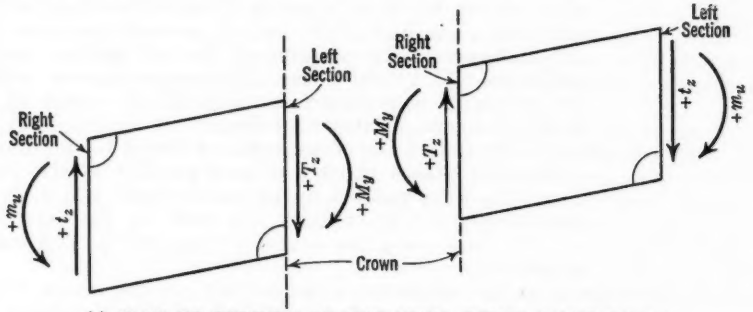
Consider the arch as fixed at the left abutment and free to move at the right abutment, that is, a curved cantilever beam fixed at the left end. When a load is applied to the arch it will deform, causing the right abutment to deflect relative to the left abutment. Now, let a force be applied at the free end so that it returns to its original position (or to any position imposed by the condition of the ends). This force must then be equal to the reaction which would be induced at the right abutment if it were prevented from deflecting under the load. The problem then becomes one of determining the amount, direction, and point of application of a force which, applied to the free end of the cantilever, will produce a deflection equal and opposite in direction to that produced by the loading.

In general, the reaction at the right abutment (which will hereafter be designated as the reaction) will be an eccentric force inclined to the plane of the abutment. This force is replaced by a force and a couple acting at the center of abutment, and then each is resolved into its components in the directions of the three major co-ordinate axes. Thus, there are six unknown quantities for the evaluation of which six simultaneous equations are required.

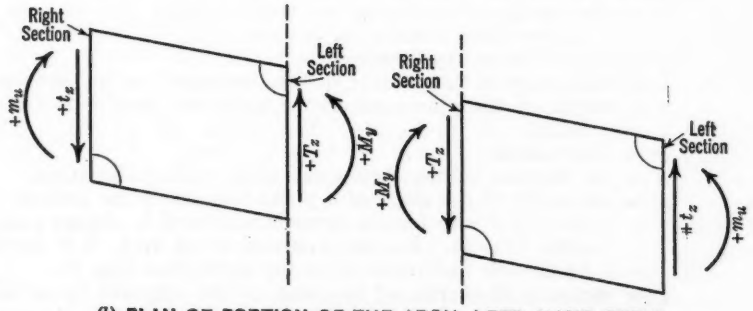
The angular deflections and linear deflections of the right abutment relative to the left abutment can each be resolved into its components in the directions of the three co-ordinate axes. These six deflection components can each be written in terms of the unknown reaction components, the external loads, and certain physical characteristics of the frame. Since the deflection due to the

loading must be equal to that due to the reaction, six equations are obtained. These apply only to the fixed arch.

For the two-hinged arch, the fact that the right abutment is hinged prevents the assumption that the angular deflections about the Z-axis are equal; and the fact that the left abutment is hinged prevents the assumption that the two deflections parallel to the Y-axis are equal. Thus, there are only four equations from statically indeterminate considerations, but two more equations can be obtained from statics. With these two additional equations, the unknown components of the reaction can be evaluated.



(*a*) PLAN OF PORTION OF THE ARCH; RIGHT-HAND SKEW



(*b*) PLAN OF PORTION OF THE ARCH; LEFT-HAND SKEW

FIG. 2.—DIAGRAM ILLUSTRATING EFFECT OF THE DIRECTION OF SKEW.

NOMENCLATURE

As previously stated, a comprehensive study of the question of signs has been made with a view of facilitating the practical application of the theory. In this derivation there are factors the signs of which are more or less awkward; hence, wherever possible, this awkwardness is thrown into those terms which appear only in the derivation, but do not occur in the final equations.

The directions of the forces and co-ordinates are all referenced to the geometric properties of the structures, rather than to directions in space. For instance, T_z, which is the horizontal shear at the crown parallel to the Z-axis, is defined as positive toward the obtuse angle of the section on which it is acting. Changing the direction of skew, changes the direction of T_z in space, but it would keep the same sign because its direction with reference to the

obtuse angle of the section would remain the same. In both Fig. 2(*a*) and Fig. 2(*b*) it should be noted that M_y produces compression at the obtuse corner of a left section and at the acute corner of a right section.

The nomenclature to be used in the derivation of the formulas is as follows:

P = any point on the neutral surface of the ring lying in that vertical plane which is half way between the spandrel walls.

x, y, z = the co-ordinates of P referred to axes through the crown for symmetrical arches, as shown in Fig. 5. The co-ordinates, x and y, are always positive, and z is positive in the direction toward the obtuse angle of the crown sections, so that when the right half of the arch is revolved about the line of symmetry to the position of the left section, the co-ordinates will be the same. For unsymmetrical arches, the origin of co-ordinates is taken at the center of the right abutment, as shown in Fig. 9.

u, v = the co-ordinates of the crown center referred to the tangent and radial planes through P and parallel to the plane, $z = 0$, u being parallel to the radial plane, and v, to the tangent plane. Their signs are fixed by the equations, $u = x \sin \phi - y \cos \phi$ and $v = x \cos \phi + y \sin \phi$, respectively.

u', v' = the u and v co-ordinates, respectively, of the center of the right abutment, and their signs are fixed by Equations (3) and (21) as given subsequently.

s = the length of an elementary parallelopiped measured on the square along the neutral surface.

h = rise of the neutral surface.

L = half span of the neutral surface measured on the square.

b = width of the ring measured parallel to the Z-axis (on the skew).

θ = skew angle.

ϵ = the tangent of the skew angle and is always positive.

ϕ = the angle of the slope of a plane tangent to the neutral surface at P and for the symmetrical arch is always positive (See Fig. 4). For the unsymmetrical arch, it is positive to the left and negative to the right (See Fig. 9).

I = moment of inertia of the area of the ring cut by a radial plane through P and parallel to the axis of the barrel (Z-axis), about a horizontal line through P and lying in the radial plane.

E = modulus of elasticity in tension and compression.

F = the factor of torsion.

λ = the ratio, $\dfrac{\text{Modulus of elasticity in compression}}{\text{Modulus of elasticity in shear}}$.

H = the horizontal load.

W = the vertical load.

x', y' = the x and y-co-ordinates of a load point (and apply only to the left half of the arch for the symmetrical arch).

T_x, T_y, T_z = the components of the crown thrust parallel, respectively, to the X-axis, Y-axis, and Z-axis. The component, T_x, is positive when it produces compression; T_y is positive away from the center of curvature (upward) on the left section, and toward the center of curvature (downward) on the right section of the crown; and T_z is positive toward the obtuse angle of the section on which it is acting.

M_x, M_y, M_z = the components of the crown moment about the X-axis, Y-axis, and Z-axis, respectively. The moment component, M_x, is positive when it tends to make the obtuse angle of the section on which it is acting move toward the center of curvature; M_y is positive when it produces compression at the obtuse angle of the left section, and tension at the obtuse angle of the right section, of the crown; and M_z is positive when it produces compression in the extrados fibers.

t_u, t_v, t_z = the components of the thrust acting on the surface cut by the radial plane through P, parallel to the u-axis, v-axis, and z-axis; t_u is positive away from the center of curvature on a left section, and toward the center of curvature on a right section; t_v is positive when it produces compression; and t_z is positive toward the obtuse angle of the section on which it is acting.

m_u, m_v, m_z = the components of the moment acting on the surface cut by a radial plane through P, about the u, v, and z-axes, respectively. The moment component, m_u, is positive when it causes compression at the obtuse angle of a left section and tension on the obtuse angle of a right section; m_v is positive when it causes the obtuse angle of the section on which it is acting to deflect toward the center of curvature, and m_z is positive when it produces compression in the extrados fibers.

R_x, R_y, R_z = the thrust components of the right reaction parallel to the X-axis, Y-axis, and Z-axis, respectively. Their signs are fixed by the relation that for $\phi = 90°$, $t_u = R_x$, $t_v = R_y$ and $t_z = R_z$; thus making R_x positive to the right; R_y, positive upward; and R_z, positive toward the obtuse angle of the right end of the structure as seen in plan.

N_x, N_y, N_z = the moment components of the right reaction about the X-axis, Y-axis, and Z-axis, respectively. Their signs are fixed by the relation that, for $\phi = 90°$, $m_u = N_x$, $m_v = N_y$, and $m_z = N_z$; thus making N_x positive when it tends to make the obtuse angle move up; N_y positive when it tends to make the obtuse angle move inward; and N_z positive when it produces compression in the extrados fibers.

$d\alpha$, $d\gamma$ = the differential angular deformation of the right face of an elementary parallelopiped relative to the left face about the v and z-axes. The positive directions of these quantities are the same as the positive directions of the moments which produce them acting on the left section; that is, m_v and m_z, respectively.

θ_x, θ_y, θ_z = the angular deflections of the center of the right abutment relative to the left abutment, about the x, y, and z-axes, respectively, and they are positive in the same directions as M_x, M_y, and M_z, acting on the left section of the crown. These symbols may have superscripts (R, H, or W), indicating whether the deflections are produced by the reactions, the horizontal loading, or the vertical loading.

Δ_x, Δ_y, Δ_z = the linear deflections of the center of the right abutment relative to the left abutment parallel to the x, y, and z-axes. The deflection, Δ_x, is positive to the right, Δ_y is positive upward, and Δ_z is positive in the same direction as the Z-axis for the right half of the arch. These symbols may also have superscripts (R, H, and W).

The factor of torsion, F, may be computed from the equation: $\frac{1}{F} = \frac{205\,\pi\,(t^2 + b^2)}{180\,b^3\,t^3}$, which may be written, $\frac{1}{F} = \frac{205\,\pi}{180}\left(\frac{1}{b^3\,t} + \frac{1}{t^3\,b}\right)$, and since b is usually large compared to t, the term, $\frac{1}{b^3\,t}$, may be neglected; hence,

$$\frac{1}{F} = \frac{205\,\pi}{180\,b\,t^3} = \frac{3.578}{b\,t^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

As the angular rotation of a shaft may be computed by the equation, $\alpha = \frac{T\,l}{E_s\,I_p}$, in which, α is the angular rotation; T, the torsion moment; l, the length of the shaft; E_s, the modulus of elasticity in shear; and I_p, the polar moment of inertia; so with a rectangular shaft the angular rotation may be computed from the formula, $\alpha = \frac{T\,l}{E_s\,F}$, in which, F has the value given in Equation (1). As the value of F enters quite prominently into the equations, reference is made to *Bulletin No. 3,* Faculty of Applied Science and Engineering, School of Engineering Research, University of Toronto, Toronto, Ont., Canada.

Mathematical Derivation of Formulas

Consider the arch as fixed at the left abutment and free to move at the right abutment. Let the arch ring be cut by a series of radial planes. These planes may be considered as cutting the ring into blocks of lengths, S, and the portions enclosed between adjacent planes as acted upon and deformed by the forces on the section between the block and the right abutment. Each block is subjected to the thrust and moment at the right abutment and the known loads, and deformed by them. The resulting deformations of each block causes the right abutment to deflect relative to the left abutment and the total is the sum of the deflections due to each block deformation.

The derivation of the formulas falls into four steps. It is necessary to find:

1.—The stresses produced at the center of the block by the loads, and the reactions at the right abutment.

2.—The deformation of the block due to these stresses.

3.—The deflections of the right abutment relative to the left abutment due to these deformations.

4(a).—For the fixed arch, these deflections produced by the reactions are equated to the deflections produced by the loads.

4(b).—For the two-hinged arch, two additional equations are obtained from statics as already explained.

In the first case considered herein, the structure is symmetrical about a vertical line through the crown. For the sake of simplicity, the origin of the principal co-ordinates is taken where this line intersects the neutral surface and will be designated as the point, C. The Y-axis is vertical, the Z-axis is parallel to the axis of the barrel, and the X-axis is normal to the other two. (See Fig. 3.)

Owing to the symmetry of the structure, the final equations simplify if written in terms of the crown stresses rather than in terms of the reactions. For this and other reasons, as will be seen later, it is more convenient to find the forces acting on a block at its center, P first in terms of the crown stresses and then to transform the resulting equations in terms of the reactions.

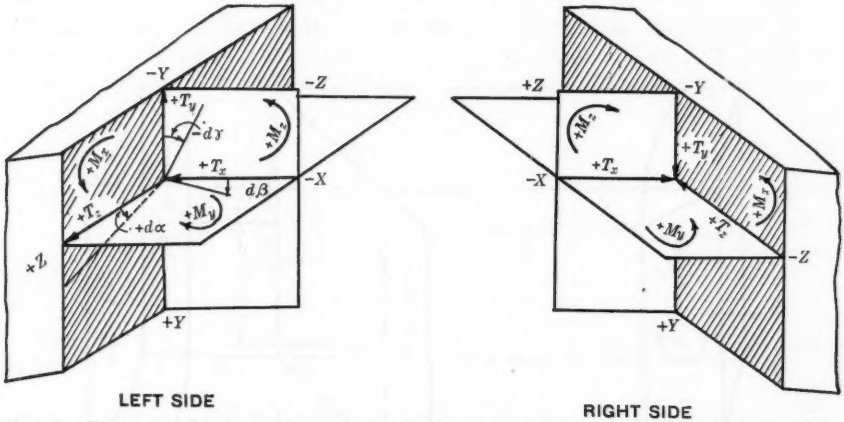


FIG. 3.—DIAGRAMS SHOWING POSITION OF CO-ORDINATE PLANES IN SYMMETRICAL ARCHES.

Cutting the arch in two parts by a radial plane through C, the half arch must be kept in equilibrium by the stresses at the crown (see Fig. 4). To find the stress at P, the section between the crown and P is taken as a free body (see Fig. 5). As it must be in equilibrium, the sum of the forces parallel to each axis and the sum of the moments about each axis must equal zero as shown in Equation (2).

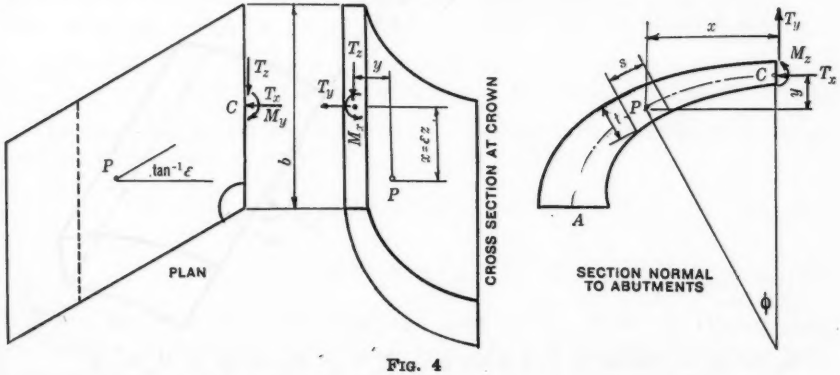


FIG. 4

Since the arch is symmetrical, for every block on the left half, there is a corresponding block on the right half. The blocks, therefore, are taken in pairs, thus reducing the algebraic work and permitting summations over one-half the arch instead of the entire arch.

Symmetrical Arches

Equations to determine the stresses at a section of a symmetrical arch may be written as follows. (See Fig. 5.) Note that for symmetrical arches the upper signs apply to the left half of the structure, and the lower signs to the right half. The loads, W and H, apply to the left half of the structure only.

$$t_v = T_x \cos \phi \mp T_y \sin \phi + W \sin \phi - H \cos \phi \quad (2a)$$

$$t_u = \pm T_x \sin \phi + T_y \cos \phi - W \cos \phi - H \sin \phi \quad (2b)$$

$$t_z = T_z \quad (2c)$$

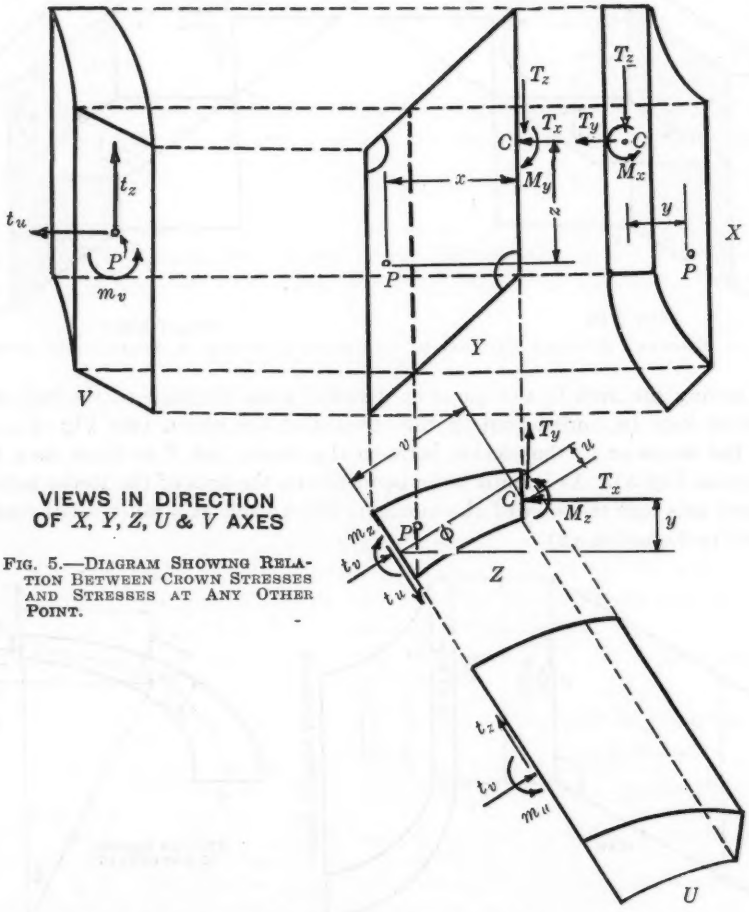


FIG. 5.—DIAGRAM SHOWING RELATION BETWEEN CROWN STRESSES AND STRESSES AT ANY OTHER POINT.

$$m_v = T_x \varepsilon x \sin \phi \pm T_y \varepsilon x \cos \phi - T_z u + M_x \cos \phi \mp M_y \sin \phi - W \varepsilon (x - x') \cos \phi - H \varepsilon (x - x') \sin \phi \quad (2d)$$

$$m_u = \mp T_x \varepsilon x \cos \phi + T_y \varepsilon x \sin \phi \pm T_z v \pm M_x \sin \phi + M_y \cos \phi - W \varepsilon (x - x') \sin \phi + H \varepsilon (x - x') \cos \phi \quad (2e)$$

and,

$$m_z = T_z y \pm T_y x + M_z - W (x - x') - H (y - y') \quad (2f)$$

Equations for the co-ordinates of the crown center, with respect to the point, P, of any section, may be written as follows:

$$u = x \sin \phi - y \cos \phi \quad \ldots\ldots (3a)$$

$$v = x \cos \phi + y \sin \phi \quad \ldots\ldots (3b)$$

and,

$$u' = x \sin \phi - y \cos \phi \pm L \sin \phi + h \cos \phi = u \pm L \sin \phi + h \cos \phi. \quad (3c)$$

These formulas, Equations (3), are restatements of the definitions given in the nomenclature. The reactions at the right abutment may be found in terms of the crown stresses by making ϕ in Equations (2) equal to 90°; thus,

$$R_x = t_u = - T_x \quad \ldots\ldots (4a)$$

$$R_y = t_v = T_y \quad \ldots\ldots (4b)$$

$$R_z = t_z = T_z \quad \ldots\ldots (4c)$$

$$N_x = m_u = T_y \varepsilon L - T_z h - M_x \quad \ldots\ldots (4d)$$

$$N_y = m_v = T_x \varepsilon L - T_z L + M_y \quad \ldots\ldots (4e)$$

and,

$$N_z = m_z = T_z h - T_y L + M_z \quad \ldots\ldots (4f)$$

The derivation of crown stresses, in terms of reactions at the right abutment, follows from Equations (4):

$$T_x = - R_x \quad \ldots\ldots (5a)$$

$$T_y = R_y \quad \ldots\ldots (5b)$$

$$T_z = R_z \quad \ldots\ldots (5c)$$

$$M_x = R_y \varepsilon L - R_z h - N_x \quad \ldots\ldots (5d)$$

$$M_y = R_x \varepsilon L + R_z L + N_y \quad \ldots\ldots (5e)$$

and,

$$M_z = R_x h + R_y L + N_z \quad \ldots\ldots (5f)$$

Stresses at a Section Produced by the Reactions at the Right Abutment.— Substituting Equations (5) in Equations 2(d) and 2(f):

$$m_z = R_x (h - y) + R_y (L \pm x) + N_z - W (x - x') - H (y - y') \quad \ldots (6a)$$

$$m_v = \mp R_x \varepsilon (L \pm x) \sin \phi + R_y \varepsilon (L \pm x) \cos \phi - R_z u' - N_x \cos \phi \mp N_y \sin \phi - W \varepsilon (x - x') \cos \phi - H \varepsilon (x - x') \sin \phi \quad \ldots\ldots (6b)$$

The equations for the deformations of an elementary parallelopiped due to stresses at its center, P, are:

$$E\, d\gamma = m_z \frac{s}{I} \quad \ldots\ldots (7a)$$

$$E\, d\alpha = m_v \frac{\lambda s}{F} \quad \ldots\ldots (7b)$$

There are four more deformations; that is, two shear, one thrust, and the bending deformation about the U-axis. As already explained these four deformations are small compared to $d\alpha$ and $d\gamma$, and will be neglected.

In Equations 8(A), Equations (6) have been rewritten by taking each reaction or load in turn and making the remaining terms zero. The equations listed, therefore, are for the stresses at a section, due to each reaction component at the right abutment.

Equations (*8A*).—Stresses at a section due to each reaction component at the right abutment, are:

Reaction component.	Left Side of Arch.		Right Side of Arch.	
	m_z.	m_v.	m_z.	m_v.
R_x	$R_x (h-y)$	$-R_x \epsilon (L+x) \sin \phi$	$R_x (h-y)$	$R_x \epsilon (L-x) \sin \phi$
R_y	$R_y (L+x)$	$+R_y \epsilon (L+x) \cos \phi$	$R_y (L-x)$	$R_y \epsilon (L-x) \cos \phi$
R_z	.....	$-R_z u'$	.....	$-R_z u'$
N_x	.....	$-N_x \cos \phi$	.....	$-N_x \cos \phi$
N_y	.....	$-N_y \sin \phi$	.....	$+N_y \sin \phi$
N_z	N_z	.....	N_z	.....

Equations (*8B*).—Stresses at a section due to a vertical load, W, at a point, $x = x'$; $z = \varepsilon x'$, are similarly given by:

$$\left.\begin{aligned} m_z &= -W (x - x') \\ \text{and,} \quad m_v &= -W \varepsilon (x - x') \cos \phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8B)$$

Equations (*8C*).—Stresses at a section, due to a horizontal load, H, at a point, $y = y'$; $z = \varepsilon x'$, are:

$$\left.\begin{aligned} m_z &= -H (y - y') \\ \text{and,} \quad m_v &= -H \varepsilon (x - x') \sin \phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8C)$$

Equations 8(*A*) give the moments produced by the individual reactions and, when combined, are used together with either Equations 8(*B*), or Equations 8(*C*), as will be shown later.

Equations (9) give the differential deflections of the right abutment relative to the left abutment, due to the deformation of a pair of corresponding elementary parallelopipeds.

Equations (*9*).—Differential deflections of the right abutment are:

Deflection.	Left Side.		Right Side.		Equation No.
	Due to m_z.	Due to m_u.	Due to m_z.	Due to m_u.	
$d\theta_x$	.....	$d\alpha \cos \phi$	.....	$d\alpha \cos \phi$	(9*a*)
$d\theta_y$	.....	$-d\alpha \sin \phi$	.....	$d\alpha \sin \phi$	(9*b*)
$d\theta_z$	$d\gamma$	.....	$d\gamma$	.....	(9*c*)
$d\Delta_x$	$d\gamma (h-y)$	$-d\alpha \epsilon (L+x) \sin \phi$	$d\gamma (h-y)$	$d\alpha \epsilon (L-x) \sin \phi$	(9*d*)
$d\Delta_y$	$d\gamma (L+x)$	$d\alpha \epsilon (L+x) \cos \phi$	$d\gamma (L-x)$	$d\alpha \epsilon (L-x) \cos \phi$	(9*e*)
$d\Delta_z$	.....	$d\alpha u'$	.....	$d\alpha u'$	(9*f*)

Differential Deflections of the Right Abutment Produced by the Reactions.—Substituting Equations (7) and 8(*A*) in Equations (9), Equation (9*a*), for example, is expanded into the form:

$$E\, d\theta_x{}^R = R_x \left[-\varepsilon (L+x) \sin \phi \cos \phi \lambda \frac{s}{F} + \varepsilon (L-x) \sin \phi \cos \phi \lambda \frac{s}{F} \right]$$

$$+ R_y \left[\varepsilon (L+x) \cos^2 \phi \lambda \frac{s}{F} + \varepsilon (L-x) \cos^2 \phi \lambda \frac{s}{F} \right]$$

$$+ R_z \left[- u'_L \cos \phi \lambda \frac{s}{F} - u'_R \cos \phi \lambda \frac{s}{F} \right] + N_x \left[- 2 \cos^2 \phi \lambda \frac{s}{F} \right]$$
$$+ N_y \left[- \sin \phi \cos \phi \lambda \frac{s}{F} + \sin \phi \cos \phi \lambda \frac{s}{F} \right] \ldots\ldots\ldots (10a)$$

The equations for the deflections corresponding to Equations (9*b*), (9*c*), etc., are similarly obtained and are not shown.

Total Deflections of the Right Abutment Produced by the Reactions (in Terms of the Crown Stresses).—Equations (11) give the total deflections of the right abutment relative to the left produced by the reactions, and are in terms of the crown stresses. They are obtained by substituting Equations (4) in Equations (10).

Making the substitutions indicated and integrating or summing up over one-half the arch:

$$E \theta_x^R = 2 \left\{ \left[\varepsilon \sum x \sin \phi \cos \phi \lambda \frac{s}{F} \right] T_x - \left[\sum u \cos \phi \lambda \frac{s}{F} \right] T_z \right.$$
$$\left. + \left[\sum \cos^2 \phi \lambda \frac{s}{F} \right] M_x \right\} \ldots\ldots\ldots\ldots\ldots\ldots (11a)$$

$$E \theta_y^R = 2 \left\{ - \left[\varepsilon \sum x \sin \phi \cos \phi \lambda \frac{s}{F} \right] T_y + \left[\sum \sin^2 \phi \lambda \frac{s}{F} \right] M_y \right\} . (11b)$$

$$E \theta_z^R = 2 \left\{ \left[\sum y \frac{s}{I} \right] T_x + \left[\sum \frac{s}{I} \right] M_z \right\} \ldots\ldots\ldots\ldots\ldots\ldots (11c)$$

$$E \Delta_x^R = 2 \left\{ - \left[\sum y^2 \frac{s}{I} + \varepsilon^2 \sum x^2 \sin^2 \phi \lambda \frac{s}{F} \right] T_x + \left[\varepsilon \sum u x \sin \phi \lambda \frac{s}{F} \right] T_z \right.$$
$$- \left[\sum y \frac{s}{I} \right] M_z - \left[\varepsilon \sum x \sin \phi \cos \phi \lambda \frac{s}{F} \right] M_x \left. \right\} + h E \theta_z^R$$
$$+ \varepsilon L E \theta_y^R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11d)$$

$$E \Delta_y^R = 2 \left\{ \left[\sum x^2 \frac{s}{I} + \varepsilon^2 \sum x^2 \cos^2 \phi \lambda \frac{s}{F} \right] T_y \right.$$
$$\left. - \left[\varepsilon \sum x \sin \phi \cos \phi \lambda \frac{s}{F} \right] M_y \right\} + L E \theta_z^R + \varepsilon L E \theta_x^R \ldots . (11e)$$

and,

$$E \Delta_z^R = 2 \left\{ \left[\varepsilon \sum u x \sin \phi \lambda \frac{s}{F} \right] T_x - \left[\sum u^2 \lambda \frac{s}{F} \right] T_z \right.$$
$$\left. + \left[\sum u \cos \phi \lambda \frac{s}{F} \right] M_y \right\} - L E \theta_y^R + h E \theta_x^R \ldots\ldots . (11f)$$

Total Deflections of the Right Abutment Produced by a Vertical Load at the Point, $x = x'$, $z = \varepsilon x'$.—Substituting Equations (7) and (8*B*) in Equations (9), and bearing in mind that a load deforms only that part of the structure between the left abutment and the load:

$$E \theta_x^W = \left[- \varepsilon \sum_A^P x \cos^2 \phi \lambda \frac{s}{F} + \varepsilon x' \sum_A^P \cos^2 \phi \lambda \frac{s}{F} \right] W \ldots\ldots\ldots (12a)$$

$$E \theta_y^W = \left[\varepsilon \sum_A^P x \sin \phi \cos \phi \lambda \frac{s}{F} - \varepsilon x' \sum_A^P \sin \phi \cos \phi \lambda \frac{s}{F} \right] W . (12b)$$

$$E\,\theta_z{}^W = \left[-\sum_A^P x\,\frac{s}{I} + x' \sum_A^P \frac{s}{I} \right] W \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12c)$$

$$E\,\Delta_x{}^W = \left[\sum_A^P x\,y\,\frac{s}{I} - x' \sum_A^P y\,\frac{s}{I} + \varepsilon^2 \sum_A^P x^2 \sin\phi\cos\phi\,\lambda\,\frac{s}{F} - \varepsilon^2\,x' \sum_A^P x \sin\phi\cos\phi\,\lambda\,\frac{s}{F} + h\,E\,\theta_z{}^W + \varepsilon\,L\,E\,\theta_y{}^W \right] W \ldots (12d)$$

$$E\,\Delta_y{}^W = \left[-\sum_A^P x^2\,\frac{s}{I} + x' \sum_A^P x\,\frac{s}{I} - \varepsilon^2 \sum_A^P x^2 \cos^2\phi\,\lambda\,\frac{s}{F} + \varepsilon^2\,x' \sum_A^P x \cos^2\phi\,\lambda\,\frac{s}{F} + L\,E\,\theta_z{}^W + \varepsilon\,L\,E\,\theta_x{}^W \right] W \ldots\ldots (12e)$$

and,

$$E\,\Delta_z{}^W = \left[-\varepsilon \sum_A^P u\,x\cos\phi\,\lambda\,\frac{s}{F} + \varepsilon\,x' \sum_A^P u\cos\phi\,\lambda\,\frac{s}{F} + h\,E\,\theta_x{}^W - L\,E\,\theta_y{}^W \right] W \ldots\ldots\ldots\ldots\ldots\ldots (12f)$$

Total Deflections of Right Abutment Produced by Horizontal Load at Point $y = y'$ and $z = \varepsilon\,x'$.—Similarly, substituting Equations (7) and (8*C*) in Equations (9):

$$E\,\theta_x{}^H = \left[-\varepsilon \sum_A^P x \sin\phi\cos\phi\,\lambda\,\frac{s}{F} + \varepsilon\,x' \sum_A^P \sin\phi\cos\phi\,\lambda\,\frac{s}{F} \right] H. \ldots (13a)$$

$$E\,\theta_y{}^H = \left[\varepsilon \sum_A^P x \sin^2\phi\,\lambda\,\frac{s}{F} - \varepsilon\,x' \sum_A^P \sin^2\phi\,\lambda\,\frac{s}{F} \right] H \ldots\ldots\ldots\ldots (13b)$$

$$E\,\theta_z{}^H = \left[-\sum_A^P y\,\frac{s}{I} + y' \sum_A^P \frac{s}{I} \right] H \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13c)$$

$$E\,\Delta_x{}^H = \left[\sum_A^P y^2\,\frac{s}{I} - y' \sum_A^P y\,\frac{s}{I} + \varepsilon^2 \sum_A^P x^2 \sin^2\phi\,\lambda\,\frac{s}{F} - \varepsilon^2\,x' \sum_A^P x \sin^2\phi\,\lambda\,\frac{s}{F} + h\,E\,\theta_z{}^H + \varepsilon\,L\,E\,\theta_y{}^H \right] H \ldots\ldots (13d)$$

$$E\,\Delta_y{}^H = \left[-\sum_A^P x\,y\,\frac{s}{I} + y' \sum_A^P x\,\frac{s}{I} - \varepsilon^2 \sum_A^P x^2 \sin\phi\cos\phi\,\lambda\,\frac{s}{F} + \varepsilon^2\,x' \sum_A^P x \sin\phi\cos\lambda\,\frac{s}{F} + L\,E\,\theta_z{}^H + \varepsilon\,L\,E\,\theta_x{}^H \right] H \ldots\ldots (13e)$$

and,

$$E\,\Delta_z{}^H = \left[-\varepsilon \sum_A^P u\,x \sin\phi\,\lambda\,\frac{s}{F} + \varepsilon\,x' \sum_A^P u \sin\phi\,\lambda\,\frac{s}{F} - L\,E\,\theta_y{}^H + h\,E\,\theta_x{}^H \right] H \ldots\ldots\ldots\ldots\ldots\ldots (13f)$$

Symmetrical Fixed Arch

If the abutments are considered fixed, all the deflections produced by a load or by temperature expansion (or contraction) are equal and opposite to the deflections produced by the reactions. (See Equations (14).)

EQUATIONS (14).—SHOWING EQUALITY OF DEFLECTIONS PRODUCED IN A FIXED ARCH BY A LOAD OR BY TEMPERATURE.

Reactions.	For vertical load.	For horizontal load.	For temperature rise.	Equation No.
$\Delta_x{}^R =$	$-\Delta_x{}^W$	$-\Delta_x{}^H$	$-2\,c\,t\,L$	(14*a*)
$\Delta_z{}^R =$	$-\Delta_z{}^W$	$-\Delta_z{}^H$	$-2\,\epsilon\,c\,t\,L$	(14*b*)
$\theta_x{}^R =$	$-\theta_x{}^W$	$-\theta_x{}^H$	0	(14*c*)
$\theta_z{}^R =$	$-\theta_z{}^W$	$-\theta_z{}^H$	0	(14*d*)
$\Delta_y{}^R =$	$-\Delta_y{}^W$	$-\Delta_y{}^H$	0	(14*e*)
$\theta_y{}^R =$	$-\theta_y{}^W$	$-\theta_y{}^H$	0	(14*f*)

Equations (14) may be rewritten as given in Equations (15). (See page 18.)

Substituting in Equation (15) the values previously obtained in Equations (11), (12), and (13), the final equations (in terms of the crown stresses) for the fixed symmetrical arch are obtained. Theses, with other final equations, are given in Appendix I, for the convenience of the designer. They are numbered Equations (16) in order to help the designer in referring back to the text when necessary.

Symmetrical Two-Hinged Arch

As in the case of a right arch, if the abutments are considered hinged,

$$\Delta_y{}^R + \Delta_y{}^H \neq 0;\ \Delta_y{}^R + \Delta_y{}^W \neq 0;\ \text{and}\ \theta_z{}^R + \theta_z{}^H \neq 0;\ \theta_z{}^R + \theta_z{}^W \neq 0$$

and two additional Equations (17) are obtained from statics which take the place of Equations (14(*d*)) and (14(*e*)).

EQUATIONS (17).—ADDITIONAL FORMULAS FOR HINGED ABUTMENTS.

	For vertical load, W.	For horizontal load, H.	For temperature rise.	Equation No.
$M_z + h\,T_x =$	$\frac{1}{2}(L - x')\,W$	$\frac{1}{2}(h - y')\,H$	0	(17*d*)
$T_y = R_y =$	$\frac{1}{2}\left(1 - \frac{x'}{L}\right)W$	$\frac{1}{2}\left(\frac{h - y'}{L}\right)H$	0	(17*e*)

Equations (14*a*), (14*b*), (14*c*), and (14*f*) still hold true; that is, these four equations with Equations (17*d*) and (17*e*), are the required six formulas and they may be rewritten as given in Equations (18). (See page 19.)

Substituting in Equations (18) the values previously obtained in Equations (11), (12), and (13), and eliminating M_z between Equations (18(*d*)) and the equation obtained by expanding Equation (18(*a*)), the final Equations (19) (in terms of the crown stresses), for the two-hinged symmetrical arches, are obtained. These are placed in the Appendix with the fixed arch equations.

EQUATIONS (15).—OBTAINED BY RE-ARRANGING EQUATIONS (14).

	For vertical load.	For horizontal load.	For temperature rise.	Equation No.
$-\frac{1}{2} E \left(\Delta_x{}^R - h\,\theta_z{}^R - \epsilon L\,\theta_y{}^R\right) =$	$+\frac{1}{2} E \left(\Delta_x{}^W - h\,\theta_z{}^W - \epsilon L\,\theta_y{}^W\right)$	$+\frac{1}{2} E \left(\Delta_x{}^H - h\,\theta_z{}^H - \epsilon L\,\theta_y{}^H\right)$	$+E\,c\,t\,L$	(15*a*)
$-\frac{1}{2} E \left(\Delta_z{}^R + L\,\theta_y{}^R - h\,\theta_x{}^R\right) =$	$+\frac{1}{2} E \left(\Delta_z{}^W + L\,\theta_y{}^W - h\,\theta_x{}^W\right)$	$+\frac{1}{2} E \left(\Delta_z{}^H + L\,\theta_y{}^H - h\,\theta_x{}^H\right)$	$+E\,\epsilon\,c\,t\,L$	(15*b*)
$+\frac{1}{2} E\,\theta_x{}^R =$	$-\frac{1}{2} E\,\theta_x{}^W$	$-\frac{1}{2} E\,\theta_x{}^H$	0	(15*c*)
$+\frac{1}{2} E\,\theta_z{}^R =$	$-\frac{1}{2} E\,\theta_z{}^W$	$-\frac{1}{2} E\,\theta_z{}^H$	0	(15*d*)
$-\frac{1}{2} E \left(\Delta_y{}^R - L\,\theta_z{}^R - \epsilon L\,\theta_x{}^R\right) =$	$+\frac{1}{2} E \left(\Delta_y{}^W - L\,\theta_z{}^W - \epsilon L\,\theta_x{}^W\right)$	$+\frac{1}{2} E \left(\Delta_y{}^H - L\,\theta_z{}^H - \epsilon L\,\theta_x{}^H\right)$	0	(15*e*)
$-\frac{1}{2} E\,\theta_y{}^R =$	$+\frac{1}{2} E\,\theta_y{}^W$	$+\frac{1}{2} E\,\theta_y{}^H$	0	(15*f*)

EQUATIONS (18).—OBTAINED BY RE-ARRANGING EQUATIONS (14) AND (17).

	For vertical, load W.	For Horizonal load, H.	For temperature rise.	Equation No.
$-\frac{1}{2} E \left(\Delta_x{}^R - \epsilon L \theta_y{}^R\right) =$	$+\frac{1}{2} E \left(\Delta_x{}^W - \epsilon L \theta_y{}^W\right)$	$+\frac{1}{2} E \left(\Delta_x{}^H - \epsilon L \theta_y{}^H\right)$	$E c t L$	(18*a*)
$-\frac{1}{2} E \left(\Delta_z{}^R + L \theta_y{}^R - h \theta_x{}^R\right) =$	$+\frac{1}{2} E \left(\Delta_z{}^W + L \theta_y{}^W - h \theta_x{}^W\right)$	$+\frac{1}{2} E \left(\Delta_z{}^H + L \theta_y{}^H - h \theta_x{}^H\right)$	$\epsilon E c t L$	(18*b*)
$+\frac{1}{2} E \theta_x{}^R =$	$-\frac{1}{2} E \theta_x{}^W$	$-\frac{1}{2} E \theta_x{}^H$	0	(18*c*)
$M_x + h T_y =$	$\frac{1}{2} (L - x') W$	$\frac{1}{2} (h - y') H$	0	(18*d*)
$T_y =$	$\frac{1}{2} \left(1 - \frac{x'}{L}\right) W$	$\frac{1}{2} \left(\frac{h - y'}{L}\right) H$	0	(18*e*)
$-\frac{1}{2} E \theta_y{}^R =$	$\frac{1}{2} E \theta_y{}^W$	$\frac{1}{2} E \theta_y{}^H$	0	(18*f*)

EXAMPLE 1.—ANALYSIS OF A TWO-HINGED SKEW FRAME

The following numerical example based on the two-hinged skew rigid frame, shown in Fig. 6, illustrates the preceding theory. Where possible, only such portions of the analysis which differ from the design of a right arch are given in detail. The work is carried through in detail to the point of obtaining influence lines, dead load, and temperature stresses. Maximum moments, shears, and thrusts are obtained in the usual manner, and the results only are given. It should be noticed, however, that in a right arch, the bending moment is most important in determining the combination of loading to produce the maximum stresses; and the thrusts are obtained for the same condition and position of loading as for the bending moment. Similarly, in a skew arch, m_z governs, and all the other stresses are obtained for the same kind of loading as is used for this moment.

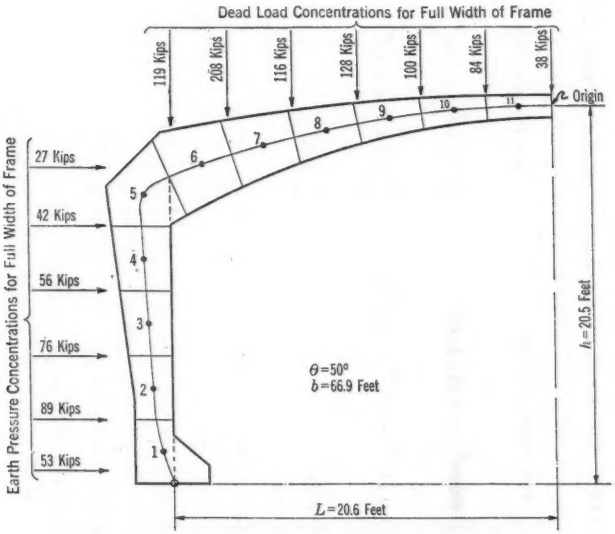


FIG. 6.—DESIGN SECTION (NORMAL TO ABUTMENTS).

Table 1 gives the elastic and geometric constants required for the evaluation of the coefficients of Equations (19).

The constants from Table 1 are substituted in the expressions for the coefficients given in Equations (19). (See Tables 2, 3, and 4.)

The unit load coefficients from Table 3 are multiplied by the dead load concentrations from Fig. 6, giving the dead load coefficients in Table 5. Similarly, the coefficients of Table 4 are multiplied by the earth pressure concentrations from Fig. 6, giving the earth pressure coefficients in Table 5.

From Tables 2, 3, and 5, the coefficients are substituted in Equations (19) which are then solved for the unknown reactions as shown in Table 6.

In practical design, Items 11, 15, 20, 25, 27, and 32 (Table 6) should be incorporated in a new and condensed table for the convenience of the

TABLE 1.—CONSTANTS.

Point.	Thickness in feet.	x.	y.	tan ϕ.	sin ϕ.	cos ϕ.	$\frac{1}{t^3}$.	$\frac{b\lambda}{F}$.	$x \sin \phi$.	$y \cos \phi$.	u.			$y^2 \frac{b}{I}$.
1	3.10	21.10	18.74	— 3.350	0.958	—0.286	0.034	0.304	20.23	—5.35	25.58			143.20
2	2.17	21.67	15.30	—16.74	0 998	—0.060	0.098	0.875	21.63	—0.92	22.55			275.00
3	2.60	21.88	11.80	—16.74	0.998	—0.060	0.057	0.509	21.83	—0.71	22.54			418.20 95.20
4	3.00	22.10	8.30	—16.74	0.998	—0.060	0.037	0.330	22.05	—0.50	22.55			513.40 30.60
5	3.95	22.05	4.78	+ 1.555	0.841	0.541	0.016	0.143	18.55	2.59	15.96			544.00 4.38
6	4.19	18.92	3.10	0.329	0.313	0.950	0.014	0.125	5.92	2.94	2.98			548.38 1.61
7	3.28	15.60	2.10	0.269	0.260	0.966	0.028	0.250	4.05	2.03	2.02			549.99 1.48
8	2.52	12.20	1.28	0.2095	0.205	0.979	0.063	0.563	2.50	1.25	1.25			551.47 1.24
9	1.92	8.76	0.65	0.1495	0.148	0.989	0.141	1.260	1.30	0.64	0.66			552.71 0.71
	1.48	5.27	0.20	0.0685	0.089	0.996	0.308	2.750	0.47	0.20	0.27			553.42 0.15
11	1.30	1.78	0.05	0.0322	0.032	1.000	0.455	4.060	0.07	0.05	0.02			553.57 0.01
														553.58

TABLE 1.—(*Continued.*)

Point.	$\sin^2\phi\frac{b\lambda}{F}$.	$x^2\sin^2\phi\frac{b\lambda}{F}$.	$u^2\frac{b\lambda}{F}$.	$u\sin\phi\frac{b\lambda}{F}$.	$u\,x\sin\phi\frac{b\lambda}{F}$.	$\frac{b}{I}$.	$x\frac{b}{I}$.	$x\,y\frac{b}{I}$.	$\cos\phi\frac{b\lambda}{F}$.	$u\cos\phi\frac{b\lambda}{F}$.
1	0.280	124.80	198.80	7.44	157.00	0.408	8.63	161.00	−0.087	−2.222
2	0.870	409.00	444.00	19.70	428.00	1.175	25.55	391.34	−0.053	−1.118
3	1.150	533.80	642.80	27.14	585.00	1.583	34.18	552.34	−0.140	−3.340
	0.506	242.00	259.00	11.47	252.00	0.684	14.99	176.74	−0.031	−0.700
4	1.656	775.80	901.80	38.61	837.00	2.267	49.17	729.08	−0.171	−4.040
	0.328	160.20	167.80	7.41	164.00	0.444	9.83	81.61	−0.020	−0.451
5	1.984	936.00	1 069.60	46.02	1 001.00	2.711	59.00	810.69	−0.191	−4.491
	0.101	49.20	36.20	1.92	42.35	0.192	4.21	20.47	0.077	1.230
6	2.085	985.20	1 105.80	47.94	1 043.35	2.903	63.21	831.16	−0.114	−3.261
	0.012	4.30	1.11		2.20	0.168	3.18	9.87	0.119	0.355
7	2.097	989.50	1 106.91		1 045.55	3.071	66.39	841.03	0.005	−2.906
	0.017	4.14	1.04		2.06	0.336	5.22	10.97	0.242	0.489
8	2.114	993.64	1 107.95		1 047.61	3.407	71.61	852.00	0.247	−2.417
	0.024	3.57	0.73		1.73	0.756	9.23	11.81	0.552	0.690
9	2.138	997.21	1 108.68		1 049.34	4.163	80.84	863.81	0.799	−1.727
	0.028	2.15	0.55		1.08	1.690	14.83	9.63	1.248	0.823
10	2.166	999.36	1 109.23		1 050.42	5.853	95.67	873.44	2.047	−0.904
	0.022	0.61	0.20		0.35	3.700	19.53	3.91	2.740	0.740
11	2.188	999.97	1 109.43		1 050.77	9.553	115.20	877.35	4.787	−0.164
	0.004	0.01	0.00		0.01	5.460	9.73	0.49	4.060	0.081
	2.192	999.98	1 109.43		1 050.78	15.013	124.93	877.84	8.847	−0.083

TABLE 1.—(*Continued.*)

2.192	999.98	1 109.43		1 050.78				

TABLE 1.—(*Continued.*)

Point.	$u\,x\cos\phi\,\frac{b\lambda}{F}$.	$\cos^2\phi\,\frac{b\lambda}{F}$.	$x\cos^2\phi\,\frac{b\lambda}{F}$.	$\sin\phi\cos\phi\,\frac{b\lambda}{F}$.	$x\sin\phi\cos\phi\,\frac{b\lambda}{F}$.	$x^2\sin\phi\cos\phi\,\frac{b\lambda}{F}$.	$y\,\frac{b}{I}$.	$\sin^2\phi\,\frac{b\lambda}{F}$.
1	−46.80	0.025	0.525	−0.083	−1.77	−37.40	7.65	5.91
2	−25.65	0.003	0.068	−0.053	−1.15	−24.95	18.00	18.90
3	−72.45	0.028	0.593	−0.136	−2.92	−62.35	25.65	24.81
	−15.30	0.002	0.040	−0.031	−0.68	−14.85	8.06	11.11
4	−87.75	0.030	0.633	−0.167	−3.60	−77.20	33.71	35.92
	− 9.97	0.001	0.026	−0.020	−0.44	− 9.76	3.68	7.28
5	−97.72	0.031	0.659	−0.187	−4.04	−86.96	37.39	43.20
	27.20	0.042	0.925	0.065	1.44	31.70	0.92	2.24
6	−70.52	0.073	1.584	−0.122	−2.60	−55.26	38.31	45.44
	6.72	0.113	2.140	0.037	0.70	13.30	0.52	
7	−63.80	0.186	3.724	−0.085	−1.90	−41.96	38.83	
	7.63	0.234	3.642	0.063	0.98	15.33	0.71	
8	−56.17	0.420	7.366	−0.022	−0.92	−26.63	39.54	
	8.42	0.540	6.590	0.113	1.38	16.81	0.97	
9	−47.75	0.960	13.956	0.091	0.46	− 9.82	40.51	
	7.20	1.232	10.800	0.185	1.62	+14.21	1.10	
10	−40.55	2.192	24.756	0.276	2.08	4.39	41.61	
	3.90	2.730	14.390	0.244	1.29	6.77	0.74	
11	−36.65	4.922	39.146	0.520	3.37	11.16	42.35	
	0.14	4.060	7.220	0.131	0.23	0.42	0.27	
	−36.51	8.982	46.366	0.651	3.60	11.58	42.62	

designer in the work that follows. This table is then used in computating Table 7 which is obtained by substituting the crown values from the condensed table in Equations (2). Table 7 should be repeated for every point or section for which an influence line is required. Figs. 7 and 8 are plotted from Table 7 and similar tables not reproduced herein.

TABLE 2.—VALUES OF THE COEFFICIENTS, K (ALL LOADINGS).
($h = 20.5$ ft.; $L = 20.6$ ft.; $b = 66.9$ ft.; $\varepsilon = 1.192$; $\varepsilon^2 = 1.420$.)

$\Sigma y^2 \frac{b}{I}$	$= 553.6$	$= 554$
$-2h\Sigma y \frac{b}{I}$	$= -2 \times 20.5 \times 42.6$	$= -1\,747$
$+h^2 \Sigma \frac{b}{I}$	$= 20.5^2 \times 15$	$= 6\,304$
$+\epsilon^2 \Sigma x^2 \sin^2\phi \frac{b\lambda}{F}$	$= 1.420 \times 1\,000$	$= 1\,420$
$\frac{b K_{11}}{s}$		$= 6\,531$
$\frac{b K_2}{s} = -\epsilon \Sigma u x \sin\phi \frac{b\lambda}{F}$	$= -1.192 \times 1\,051$	$= -1\,253$
$\frac{b K_3}{s} = \epsilon \Sigma x \sin\phi \cos\phi \frac{b\lambda}{F}$	$= 1.192 \times 3.60$	$= 4.291$
$\frac{b K_5}{s} = \Sigma u^2 \frac{b\lambda}{F}$	$= 1\,109$	$= 1\,109$
$\frac{b K_6}{s} = -\Sigma u \cos\phi \frac{b\lambda}{F}$	$= (-0.083)$	$= +0.083$
$\frac{b K_7}{s} = \Sigma \cos^2\phi \frac{b\lambda}{F}$	$= 8.982$	$= +8.982$
$-\Sigma \sin^2\phi \frac{b\lambda}{F}$	$= -2.192$	$= -2.192$
$-\Sigma y \frac{b}{I}$	$= -42.6$	$= -42.6$
$+h\Sigma \frac{b}{I}$	$= 20.5 \times 15$	$= 307.5$
$\frac{b K_{10}}{s}$		$= +264.9$

If these curves should reach zero or unity at the point, $x = 20.6$, they do not do so for points along the abutment, because of a negligible approximation in the computation.

In a frame of the type shown in Fig. 6, the two critical sections are the crown and the springing line. It is customary to design these two sections first to test the adequacy of the assumed frame. If these two sections are about the right dimensions (small changes being unimportant), other sections are designed in order to determine the reinforcing steel patterns. Table 8 gives the total stresses for the two critical sections. The right half of the

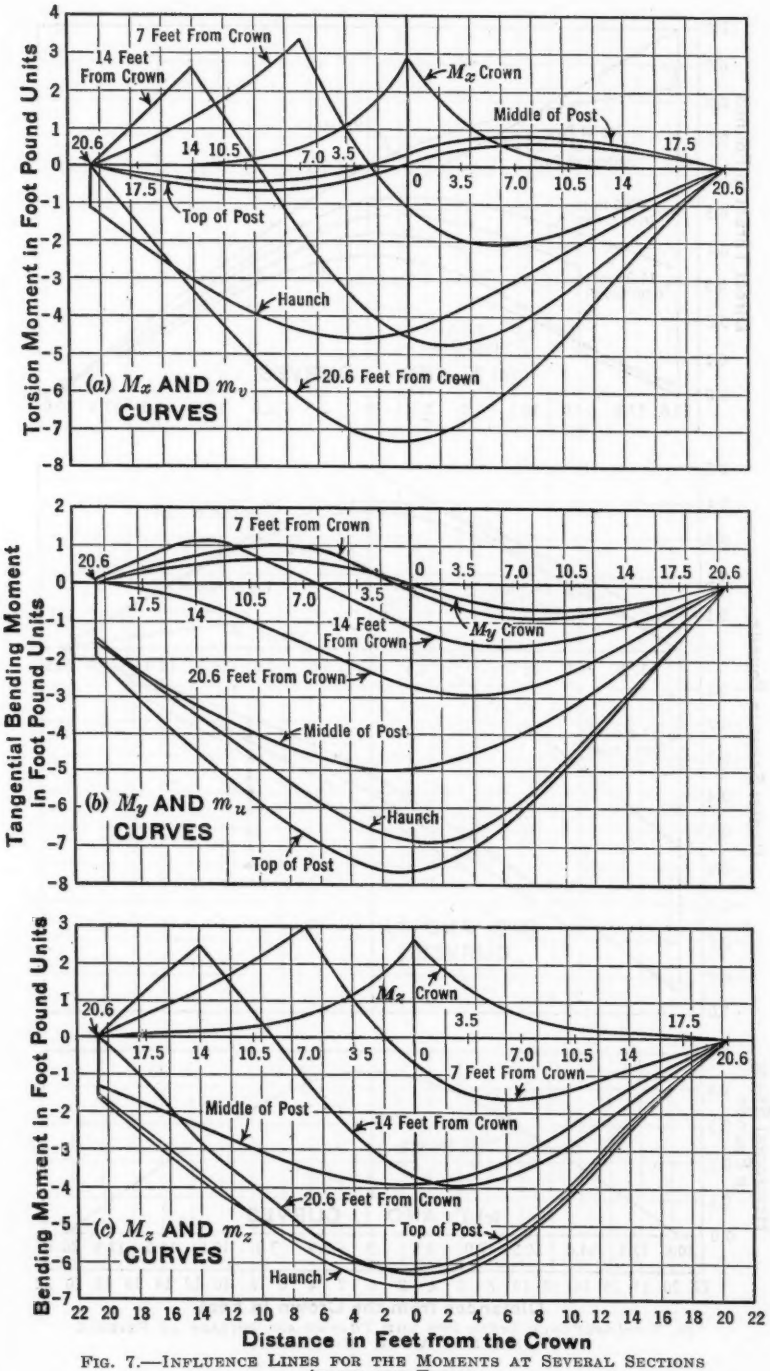

Torsion Moment in Foot Pound Units
7 Feet From Crown
14 Feet From Crown
M_x Crown
Middle of Post
Top of Post
Haunch
20.6 Feet From Crown
(a) M_x AND m_v CURVES
Tangential Bending Moment in Foot Pound Units
M_y Crown
(b) M_y AND m_u CURVES
Bending Moment in Foot Pound Units
M_z Crown
(c) M_z AND m_z CURVES
Distance in Feet from the Crown

FIG. 7.—INFLUENCE LINES FOR THE MOMENTS AT SEVERAL SECTIONS ALONG THE FRAME.

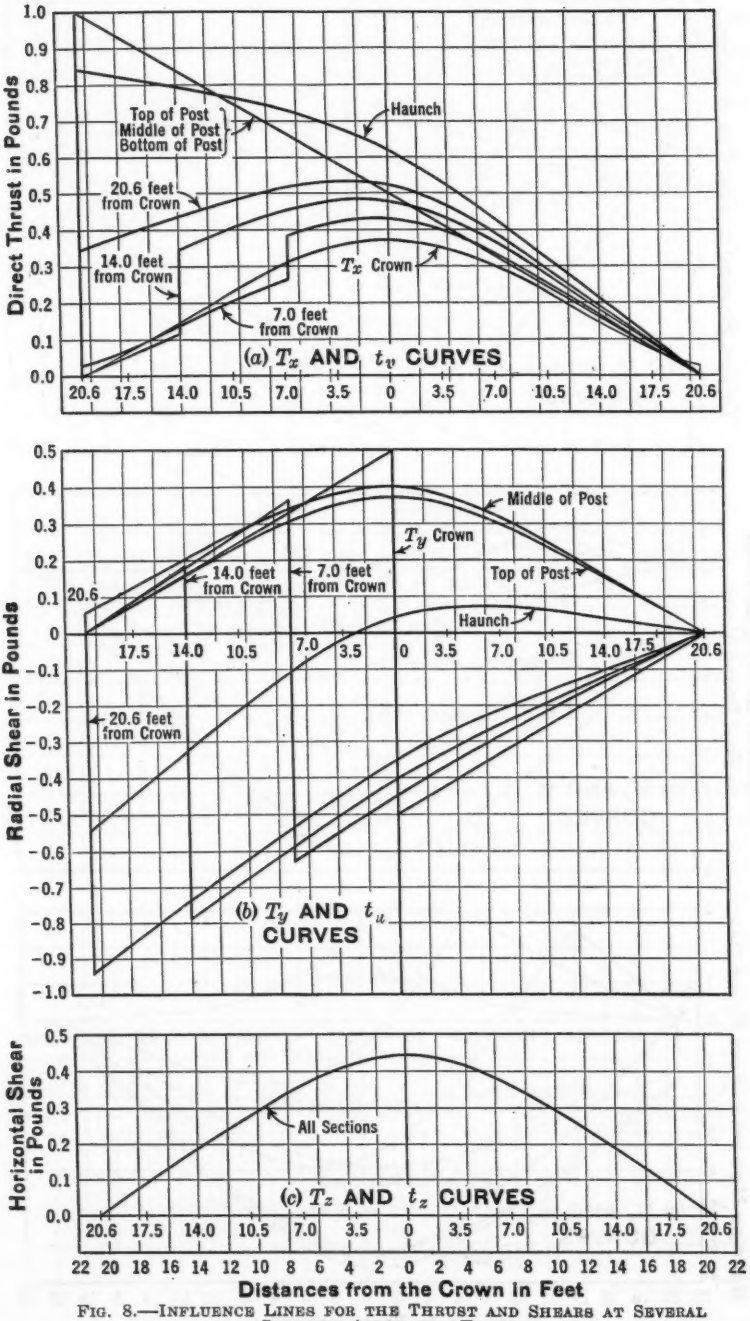


FIG. 8.—INFLUENCE LINES FOR THE THRUST AND SHEARS AT SEVERAL SECTIONS ALONG THE FRAME.

TABLE 3.—VALUES OF THE COEFFICIENTS, C, FOR VERTICAL LOADING.

Item.	$x' =$	20.6		17.5		14.0	
2	$(L-x')\left(-\sum y\frac{b}{I}+h\sum\frac{b}{I}\right)=$	0	= 0	(3.1) (265)	= 821.5	(6.6) (265)	= 1 749
3	$+\sum_A^P x\,y\,\frac{b}{I}=$	+831	= + 831	+841	= + 841	+852	= + 852
4	$-x'\sum_A^P y\,\frac{b}{I}=$	−(20.6) (38.3)	= − 789	−(17.5) (38.8)	= − 679	−(14.0) (39.5)	= − 553
5	$-h\sum_A^P x\,\frac{b}{I}=$	−(20.5) (63.2)	= −1 295	−(20.5) (66.4)	= −1 361	−(20.5) (71.6)	= −1 468
6	$+h\,x'\sum_A^P\frac{b}{I}=$	(20.5) (20.6) (2.90)	= 1 225	(20.5) (17.5) (3.07)	= 1 101	(20.5) (14.0) (3.41)	= 978.7
7	$+\epsilon^2\sum_A^P x^2\sin\phi\cos\phi\,\frac{b\,\lambda}{F}=$	+(1.420) (−55.3)	= − 78.53	(1.420) (−42.0)	= − 59.64	(1.420) (−26.6)	= − 37.77
8	$-\epsilon^2\,x'\sum_A^P x\sin\phi\cos\frac{b\,\lambda}{F}=$	−(1.420) (20.6) (−2.60)	= 76.06	−(1.420) (17.5) (−1.9)	= 47.22	−(1.420) (14.0) (−0.92)	= 18 29
9	$2\,C_{11}\,\frac{b}{s}=$		− 30.47		711.08		1 539.22

TABLE 3.—(*Continued.*)

Item.	x' =		20.6		17.5		14.0	
10	$-\sum_A^P u\, x \cos\phi \frac{b\lambda}{F}$	=	−(−70.5) =	70.5	−(−63.8) =	63.8	−(−56.2) =	56.2
11	$+x'\sum_A^P u \cos\phi \frac{b\lambda}{F}$	=	(20.6) (−3.26) =	−67.16	(17.5) (−2.91) =	−50.93	(14.0) (−2.42) =	33.88
12	$2\,C_2 \frac{b}{s}$	=	3.98 =	ϵ (3.34)	15.35 =	ϵ (12.87)	26.6 =	ϵ (22.32)
13	$+\sum_A^P x \cos^2\phi \frac{b\lambda}{F}$	=	1.584 =	1.584	3.72 =	3.72	=	7.37
14	$-x'\sum_A^P \cos^2\phi \frac{b\lambda}{F}$	=	−(20.6) (0.073) =	−1.504	−(17.5) (0.186) =	−3.255	−(14) (0.43) =	−5.88
15	$2\,C_3 \frac{b}{s}$	=	0.09534 =	ϵ (0.080)	0.554 =	ϵ (0.465)	1.776 =	ϵ (1.49)
16	$2\,C_{14} = L - x'$	=		0		3.1		6.6
17	$2\,C_{15} = 1 - \frac{x'}{L}$	=	1 − 1 =	0	1 − 0.850 =	0.150	1 − 0.680 =	0.320
18	$\sum_A^P x \sin\phi \cos\phi \frac{b\lambda}{F}$	=	(−2.60) =	−2.60	(−1.90) =	−1.90	(−0.92) =	−0.92
19	$-x'\sum_A^P \sin\phi \cos\phi \frac{b\lambda}{F}$	=	−(20.6) (−0.122) =	2.513	−(17.5) (−0.085) =	1.488	−(14) (−0.022) =	0.308
20	$2\,C_6 \frac{b}{s}$	=	−0.1037 =	ϵ (−0.087)	−0.4911 =	ϵ (−0.412)	−0.728 =	ϵ (−0.612)

TABLE 3.—(*Continued.*)

Item.	10.5		7.0		3.5		0.0	
1								
2	(10.1) (265)	= 2 667	(13.6) (265)	= 3 604	(17.1) (265)	= 4 532	(20.6) (265)	= 5 459
3	864	= 864	873	= 873	877	= 877	878	= 878
4	−(10.5) (40.5)	= − 425.3	−(7) (41.61)	= − 291.3	−(3.5) (42.4)	= − 148.4		
5	−(20.5) (80.8)	= −1 656	−(20.5) (95.7)	= − 1 962	−(20.5) (115.2)	= −2 362	−(20.5) (124.9)	= −2 560
6	(20.5) (10.5) (4.16)	= 895.4	(20.5) (7.0) (5.85)	= 839.5	(20.5) (3.5) (9.55)	= 685.2		
7	(1.420) (−9.82)	= − 13.94	(1.420) (4.39)	= 6.234	(1.420) (11.16)	= 15.85	(1.420) (11.58)	= 16.44
8	−(1.420) (10.5) (+0.46)	= − 6.85	−(1.420) (7.0) (2.08)	= − 20.68	−(1.420) (3.5) (3.37)	= 16.75		
9		2 334.31		3 048.754		3 582.90		3 793.44
10	−(−47.8)	= 47.8	−(−40.6)	= 40.6	−(−36.7)	= 36.7	−(−36.5)	= 36.5
11	(10.5) (−1.727)	= − 18.1	(7) (−0.904)	= − 6.3	(3.5) (−0.164)	= − 0.6		
12		35.4 = ϵ (29.7)		40.9 = ϵ (34.3)		43.1 = ϵ (36.1)		43.5 = ϵ (36.5)
13	13.96	= 13.96	24.76	= 24.76	39.1	= 39.1	46.4	= 46.4
14	−(10.5) (0.96)	= − 10.08	−(7) (2.19)	= − 15.33	−(3.5) (4.92)	= − 17.22		
15		4.62 = ϵ (3.88)		11.24 = ϵ (9.43)		26.08 = ϵ (21.88)		55.3 = ϵ (46.4)
16		10.1		13.6		17.1		20.6
17	1−0.510	= 0.490	1−0.340	= 0.660	1−0.170	= 0.830	1 − 0	= 1.000
18	(+0.46)	= + 0.46	2.08	= 2.08	3.37	= 3.37	3.60	= 3.60
19	−(10.5) (0.091)	= − 0.9555	−(7) (0.276)	= − 1.932	−(3.5) (0.520)	= − 1.82		
20		−0.5906 = ϵ (−0.4955)		0.1764 = ϵ (+0.148)		1.847 = ϵ (1.55)		4.290 = ϵ (3.60)

TABLE 4.—VALUES OF COEFFICIENTS, C, FOR HORIZONTAL LOADING.

Item.	x', y' =	$x' = 22.6$, $y' = 20.5$	$x' = 22.7$, $y' = 17.0$	$x' = 23.0$, $y' = 13.55$
1				
2	$(h - y')\left(-\sum y \frac{b}{I} + h \sum \frac{b}{I}\right) =$		$+(3.5)\ (+265) = 927.5$	$(6.95)\ (265) = 1\,841.8$
3	$+\sum_A^P y^2 \frac{b}{I} =$		$143.2 = 143.2$	$418.0 = 418.0$
4	$-y' \sum_A^P y \frac{b}{I} =$		$-(17)\ (7.65) = -130.1$	$-(13.55)\ (25.7) = -348.2$
5	$-h \sum_A^P y \frac{b}{I} =$		$-(20.5)\ (7.65) = -156.8$	$-(20.5)\ (25.7) = -526.8$
6	$+h y' \sum_A^P \frac{b}{I} =$		$(20.5)\ (17)\ (0.408) = 142.2$	$(20.5)\ (13.55)\ (1.583) = 440.0$
7	$+\epsilon^2 \sum_A^P x^2 \sin^2\phi \frac{b\lambda}{F} =$		$(1.420)\ (124.8) = 177.2$	$(1.420)\ (534) = 758.3$
8	$-\epsilon^2 x' \sum_A^P x \sin^2\phi \frac{b\lambda}{F} =$		$-(1.420)\ (5.91)\ (22.7) = -190.5$	$-(1.420)\ (23.0)\ (24.8) = -810.0$
9	$2\,C_{11} \frac{b}{s} =$	0	912.7	1 773.1

TABLE 4.—(*Continued.*)

Item.	$x', y' =$	$x' = 22.6, y' = 20.5$	$x' = 22.7, y' = 17.0$	$x' = 23.0, y' = 13.55$
10	$-\sum_A^P u\, x \sin\phi \frac{b\lambda}{F} =$		$-(157) = -157$	$-(585) = -585$
11	$+x'\sum_A^P u \sin\phi \frac{b\lambda}{F} =$		$(22.7)\ (7.44) = 168.9$	$(23.0)\ (27.14) = 624$
12	$2\,C_2 \frac{b}{s} =$	0	$14.2 = \epsilon\ (11.9)$	$46.5 = \epsilon\ (39)$
13	$\sum_A^P x \sin\phi \cos\phi \frac{b\lambda}{F} =$		$(-1.77) = -1.77$	$(-2.92) = -2.92$
14	$-x'\sum_A^P \sin\phi \cos\phi \frac{b\lambda}{F} =$		$-(-0.083)\ (22.7) = 1.88$	$-(-0.136)\ (23.0) = 3.13$
15	$2\,C_3 \frac{b}{s} =$	0	$0.131 = \epsilon\ (0.11)$	$0.250 = \epsilon\ (0.21)$
16	$2\,C_{14} = h - y' =$	M_x for total earth pressure can be computed directly.		
17	$2\,C_{15} = \frac{h-y'}{L} =$	T_y for total earth pressure can be computed directly; also $T_y = 0$ for symmetrical loading.		
18	$\sum_A^P x \sin^2\phi \frac{b\lambda}{F} =$			
19	$-x'\sum_A^P \sin^2\phi \frac{b\lambda}{F} =$	Need not be computed for symmetrical loading since $M_y = 0$.		
20	$2\,C_6 \frac{b}{s} =$			

TABLE 4.—(*Continued.*)

Item.	$x' = 23.4$, $y = 10.0$			$x' = 23.8$, $y' = 6.54$			$x' = 23.0$, $y' = -1.34$		
1									
2	(10.5) (265)	=	2 782.5	(13.96) (265)	=	3 699.4	(21.8) (265)	=	5 780
3	(513.4)	=	513.4	544	=	544.0	548	=	548
4	(10.0) (33.71)	=	337.1	−(6.54) (37.4)	=	244.6	−(−1.34) (38.3)	=	51
5	−(20.5) (33.71)	=	691.0	−(20.5) (37.4)	=	766.7	−(20.5) (38.3)	=	782
6	+(20.5) (10.0) (2.27)	=	466.0	+(20.5) (6.54) (2.71)	=	364.0	+(20.5) (−1.34) (2.90)	=	80
7	+(1.420) (776)	=	1 102.0	+(1.420) (986)	=	1 329.0	+(1.420) (985)	=	1 400
8	−(1.420) (35.92) (23.4)	=	−1 192.0	−(1.420) (43.2) (23.8)	=	−1 460.0	−(1.420) (23.0) (45.4)	=	−1 482
9			2 643.8			3 465.1			5 435
10	−(887)	=	− 837	−(1 001)	=	−1 001	−(1 043)	=	−1 043
11	(23.4) (38.6)	=	903.2	(28.8) (46.0)	=	1 094.8	(23.0) (47.9)	=	1 102
12		78.9	= ϵ (66.2)		111.8	= ϵ (93.8)		70.3	= ϵ (59)
13	{−(3.60)	=	− 3.60	{−(4.04)	=	− 4.04	{−(2.60)	=	− 2.60
14	−(−0.167) (23.4)	=	3.91	−(−0.187) (23.8)	=	4.45	−(−0.122) (23.0)	=	2.81
15		0.369	= ϵ (0.31)		0.488	= ϵ (0.41)		0.25	= ϵ (0.21)
16									
17									
18									
19									
20									

table is carried out only far enough to determine the possibility of a reversal of stresses. In practice, Table 8 is continued to include every section for which an influence line is drawn.

TABLE 5.—DEAD LOAD, EARTH PRESSURE, AND TEMPERATURE COEFFICIENTS.

x' or y'.	Load, in kips.	$2\,C_{11}\frac{b}{s}$.		$2\,C_2\frac{b}{s}$.		$2\,C_3\frac{b}{s}$.	
		Unit load.	Dead load.	Unit load.	Dead load.	Unit load.	Dead load.
DEAD LOAD.							
20.6	119	−30.5	−3 630	3.98	473	0.0953	11.3
17.5	208	711	148 000	15.35	3 200	0.554	115.0
14.0	166	1 539	255 500	26.6	4 420	1.776	295
10.5	128	2 334	298 500	35.4	4 530	4.62	592
7.0	100	3 048	304 800	40.9	4 090	11.24	1 124
3.5	84	3 583	301 000	43.1	3 620	26.08	2 190
0	38	3 793	144 000	43.5	1 650	55.3	2 100
	843		1 448 170		21 983		6 427
EARTH PRESSURE.							
19.8	53	183*	9 720	3 *	159	0.026*	1.6
17.0	89	913	81 600	14.2	1 263	0.131	11.7
13.55	76	1 773	135 000	46.5	3 540	0.250	19.0
10.0	56	2 644	148 000	78.9	4 420	0.369	20.7
6.54	42	3 465	145 800	111.8	4 680	0.488	20.5
3.34	27	4 340*	107 200	86.3*	2 330	0.340*	9.2
	343		627 320		16 392		82.7

TEMPERATURE (FOR 100° RISE)

$$C_{11}\frac{b}{s} = \frac{b}{s}\,E\dagger\,c\,t\,L = \frac{66.9}{3.52} \times 432\,000 \times 0.0000065 \times 100 \times 20.6 = 110\,000$$

$$C_2\frac{b}{s} = e\,C_{11}\frac{b}{s} = e \times 110\,000 = 131\,500$$

$$C_3 = C_{14} = C_{15} = C_6 = 0$$

*These values are interpolated.

† For $n = 10$, $E_c = 30\,000$ kips per sq. in. $\times 10 = 432\,000$ kips per sq. ft.

It should be noted that the weight of the abutments has been omitted from the computations because the influence lines for the crown stresses are all zero over the reactions. In computing the direct thrust, t_v, for sections along the abutment, the weight of the wall should be included. Having the maximum stresses, the sections can now be proportional. This portion of the numerical work will be given in Part II, Example 2, where the "Theory of Proportioning Sections" is discussed.

TABLE 6.—SOLUTION OF EQUATIONS.

Item.		Values of x'.							Earth pressure.	Dead load.	Temperature, 100° rise.
		0	3.5	7	10.5	14	17.5	20.6			
1	$6\,531\ T_x - 1\,253\ T_z + 4.291\ M_x =$	1 897	1 792	1 524	1 167	770	355	−15.24	627 000*	1 448 000*	110 000
2	$-1\,253\ T_x + 1\,109\ T_z + 0.083\ M_x =$	21.7	21.6	20.5	17.7	13.3	7.67	1.99	16 390*	22 000*	131 500
3	$4.291\ T_x + 0.083\ T_z + 8.982\ M_x =$	27.65	13.04	5.62	2.31	0.888	0.277	0.0477	82.7*	6 430*	0
4	$2.050\ T_x + 0.0398\ T_z + 4.291\ M_x =$	13.21	6.229	2.684	1.103	0.424	0.132	0.0228	39.6	3 070	0
5	$0.0397\ T_x + 0.00077\ T_z + 0.083\ M_x =$	0.257	0.12	0.052	0.0213	0.0082	0.0025	0.0004	0.763	57	0
6	$6\,529\ T_x - 1\,253\ T_z =$	1 884	1 786	1 521	1 166	769.6	355.4	−15.22	627 000	1 445 000	110 000
7	$-1\,253\ T_x + 1\,109\ T_z =$	21.4	21.5	20.5	17.7	13.3	7.67	1.99	16 390	22 000	131 500
8	$5.211\ T_x - T_z =$	1.504	1.425	1.214	0.931	0.614	0.284	− 0.012	498	1 152	87.8
9	$-1.129\ T_x + T_z =$	0.020	0.020	0.019	0.016	0.012	0.007	0.002	15	20	118.6
10	$4.082\ T_x =$	1.524	1.445	1.233	0.947	0.626	0.291	−0.010	513	1 172	206.4
11	$T_z =$	0.373	0.353	0.302	0.232	0.153	0.071	−0.0034	125	287	50.5
12	$5.883\ T_x - 1.129\ T_z =$	1.699	1.609	1.370	1.051	0.707	0.329	−0.0137	563	1 302	99
13	$-5.883\ T_x + 5.211\ T_z =$	0.104	0.104	0.094	0.083	0.063	0.036	0.0094	78	104	618
14	$4.082\ T_x =$	1.803	1.713	1.464	1.134	0.770	0.365	−0.0043	641	1 406	717
15	$T_z =$	0.442	0.420	0.359	0.277	0.189	0.089	−0.001	157	344	176
16	$4.291\ T_x =$	1.601	1.519	1.296	0.996	0.657	0.305	− 0.0146	537	1 232	217
17	$+ 0.083\ T_z =$	0.037	0.035	0.030	0.023	0.016	0.007	0.000	13	28	15
18	$4.291\ T_x + 0.083\ T_z =$	1.638	1.554	1.326	1.019	0.673	0.312	−0.0146	550	1 260	232

* These values are for the entire arch loaded, and equal $2\ C_{11}\frac{b}{s}$, $2\ C_2\ \frac{b}{s}$, etc.

* These values are for the entire arch ribbed, [illegible]

TABLE 6—(*Continued*).

Item.			Values of x'.							Earth pressure.	Dead load.	Temperature, 100° rise.
			0	3.5	7	10.5	14	17.5	20.6			
19	$8.982\ M_x$	=	26.0	11.49	4.29	1.29	0.215	−0.035	0.062	−467	5 170	− 232
20	M_x	=	2.89	1.28	0.48	0.14	0.02	0	0	− 52	575	− 26
21												
22												
23	$M_x + 20.5\ T_x$	=	10.30	8.55	6.80	5.05	3.30	1.55	0			0
24	$20.5\ T_x$	=	7.65	7.26	6.19	4.76	3.14	1.46	−0.07			1 035
25	M_x	=	2.65	1.29	0.61	0.29	0.16	0.09	+0.07	−60 *	860*	− 1 035
26												0
27	T_y	=	0.500	0.415	0.330	0.245	0.160	0.075	0	0	0	
28												
29	$-2.192\ M_y + 4.291\ T_y$	=	2.15	0.924	0.088	−0.300	−0.364	−0.250	−0.052	0	0	0
30	$4.291\ T_y$	=	2.15	1.780	1.416	1.051	0.687	0.322	0	0	0	0
31	$-2.192\ M_y$	=	0	−0.856	−1.328	−1.351	−1.051	−0.572	−0.052	0	0	0
32	M_y	=	0	0.391	0.606	0.616	0.479	0.261	0.024	0	0	0

* For earth pressure and dead load, M_x is best computed directly (from considerations of static equilibrium) and is so computed in this table. The computation is not shown.

TABLE 7.—STRESSES 14 FEET 0 INCHES FROM CROWN.

($x = 13.87$; $y = 1.60$; $\sin\phi = 0.233$; $\cos\phi = 0.973$; $\varepsilon = 1.192$; $\varepsilon \sin\phi = 0.277$; $\varepsilon \cos\phi = 1.159$; $\varepsilon x \sin\phi = 3.86$; $\varepsilon x \cos\phi = 16.08$; $u = 1.67$; $v = 13.87$; $w = 433K$; $\bar{x}' = 8.00$.)

x'.	0	3.5	7.0	10.5	14.0	17.5	20.6	Dead load.	Earth pressure.	Temperature.
$T_x \cos\phi$	0.363	0.344	0.294	0.226	0.150	0.069	0	281	122	49
$\mp T_y \sin\phi$	$\mp$ 0.116	$\mp$ 0.096	$\mp$0.077	$\mp$0.057	$\mp$0.035	$\mp$0.017	0	0	0	0
$W \sin\phi$	+ 0.233	0.233	0.233	0.233	0.233	0	0	101	0	0
t_v (left)	0.480	0.481	0.450	0.402	0.348	0.052	0	382	122	49
t_v (right)	0.479	0 440	0.371	0.283	0.185	0.086	0	382	122	49
$\pm T_x \sin\phi$	$\pm$ 0.087	$\pm$ 0.082	$\pm$0.070	$\pm$0.054	$\pm$0.036	$\pm$0.016	0	$\pm$ 72	$\pm$ 29	$\pm$ 12
$+ T_y \cos\phi$	0.486	0.404	0.321	0.238	0.146	0.072	0	0	0	0
$- W \cos\phi$	− 0.973	− 0.973	−0.973	−0.973	−0.973	0	0	$\mp$ 422	0	0
t_u (l)	− 0.400	− 0.487	−0.582	−0.681	−0.791	+0.088	0	− 350	+ 29	+ 12
t_u (r)	0.399	0.322	0.251	0.184	+0.182	+0.056	0	+ 350	− 29	− 12
$t_z = T_z$	0.441	0.419	0.360	0.278	0.186	0.088	0	346	160	176
$T_x y$	0.597	0.566	0.483	0.371	0.246	0.117	0	462	200	80
$\pm T_y x$	$\pm$ 6.935	$\pm$ 5.756	$\pm$4.577	$\pm$3.398	$\pm$2.081	$\pm$1.026	0	0	0	0
M_x	2.640	1.280	0.590	0.290	0.150	0.080	0	820	− 51	−1 035
$- W (x - x')$	−13.870	−10.370	−6.870	−3.370	0	0	0	−2 542	0	0
m_x (l)	− 3.698	− 2.748	−1.220	0.689	2.477	1.223	0	−1 260	149	− 955
m_x (r)	− 3.698	− 3.910	− 3.504	−2.737	−1.685	−0.829	0	−1 260	149	− 955
$\mp T_x \epsilon x \cos\phi$	$\mp$ 5.996	$\mp$ 5.691	$\mp$4.855	$\mp$3.730	$\mp$2.476	$\mp$1.130	0	$\mp$4 641	$\mp$2 010	$\mp$ 812
$+ Ty \epsilon x \sin\phi$	1.918	1.592	1.266	0.940	0.575	0.284	0	0	0	0
$\pm T_z v$	$\pm$ 6.117	$\pm$ 5.812	$\pm$4.993	$\pm$3.856	$\pm$2.579	$\pm$1.221	$\pm$0.083	$\pm$4 799	$\pm$2 219	$\pm$2 434
$\pm M_x \sin\phi$	$\pm$ 0.673	$\pm$ 0.297	$\pm$0.111	$\pm$0.033	$\pm$0.005	$\pm$0	$\pm$0	$\pm$ 133	$\mp$ 12	$\mp$ 7
$M_y \cos\phi$	0	0.380	0.585	0.622	0.447	0.250	0.023	0	0	0
$- W \epsilon (x - x') \sin\phi$	− 3.842	− 2.872	−1.903	−0.935	0	0	0	$\mp$ 704	0	0
m_u (l)	− 1.130	− 0.482	−0.197	+0.786	+1.130	0.626	0.106	− 413	197	1 615
m_u (r)	1.130	1.554	1.572	1.403	0.914	0.443	−0.060	413	− 197	−1 615
$T_x \epsilon x \sin\phi$	1.431	1.358	1.158	0.890	0.591	0.274	0	1 107	479	193
$\pm T_y \epsilon x \cos\phi$	$\pm$ 8.038	$\pm$ 6.672	$\pm$5.305	$\pm$3.939	$\pm$2.411	$\pm$1.190	0	0	0	0
$- T_z u$	− 0.736	− 0.700	−0.601	−0.464	−0.310	−0.147	0	− 578	− 267	− 293
$M_x \cos\phi$	2.811	1.242	0.462	0.138	0.022	0	0	556	− 52	− 26
$\mp M_y \sin\phi$	0	$\mp$ 0.091	$\mp$0.140	$\mp$0.149	$\mp$0.107	$\mp$0.060	0	0	0	0
$- W \epsilon (x - x') \cos\phi$	−16.076	−12.018	−7.962	−3.906	0	0	0	−2 940	0	0
m_v (l)	− 4.532	− 3.537	−1.798	0.448	+2.607	1.257	0	−1 860	160	− 126
m_v (r)	− 4.532	− 4.681	−4.146	−3.226	− 2.101	−1.003	0	−1 860	160	− 126
$x - x'$	13.87	10.37	6.87	3.37	0			5.87		

TABLE 8.—TOTAL STRESSES.

	Loading.	COMBINATION FOR MAXIMUM TOTAL STRESSES.						COMBINATION FOR MINIMUM TOTAL STRESSES.					
		M_z.	M_y.	T_x.	M_x.	T_y.	T_z.	M_z.	M_y.	T_x.	M_x.	T_y.	T_z.
Crown	Dead loads	+ 860	0	+ 289	+ 572	0	+ 346	+ 860					
	Earth pressures	− 60	0	+ 125	− 53	0	+ 160	− 60					
	Live loads	+ 192	± 7	+ 29	+ 209	± 38	+ 34	....					
	Total	+ 992	....	+ 443	+ 728	....	+ 540	+ 800					
	Temperature, 50°	+ 517	....	− 25	+ 13	....	− 88	− 517					
	Total	+1 509	....	+ 418	+ 741	....	+ 452	+ 283					
20.6 ft. from crown		m_z.	m_u.	t_v.	m_v.	t_u.	t_z.	m_z.	m_u.	t_v.	m_v.	t_u.	t_z.
	Dead loads	−4 870	−2 071	+ 558	−5 848	− 695	+ 346	−4 870					
	Earth pressures	+ 405	+ 393	+ 118	+ 422	+ 42	+ 160	+ 405					
	Live loads	−1 035	− 432	+ 103	−1 200	− 110	+ 73	....					
	Total	−5 500	−2 110	+ 779	−6 626	− 763	+ 579	−4 465					
	Temperature, 50°	− 425	+1 231	+ 24	− 115	+ 9	+ 88	+ 425					
	Total	−5 925	− 879	+ 803	−6 741	− 754	+ 667	−4 040					

UNSYMMETRICAL ARCHES

It is usually advantageous to take the origin either at the movable end of the transformed structure, or at the point of symmetry. The origin is, therefore, taken at the right abutment which is considered the movable end. (See Fig. 9.)

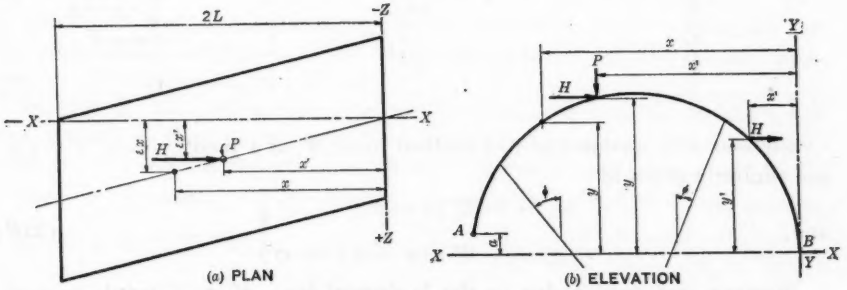


FIG. 9.—DIAGRAMS SHOWING POSITION OF CO-ORDINATE AXES FOR UNSYMMETRICAL ARCHES.

The stresses at a section of an unsymmetrical arch in terms of the reactions may be found by substituting Equations (5) in Equations (2), and transferring the origin to the center of the right abutment:

$$t_v = -R_z \cos\phi - R_y \sin\phi + W \sin\phi - H \cos\phi \quad \ldots\ldots (20a)$$

$$t_u = -R_x \sin\phi + R_y \cos\phi - W \cos\phi - H \sin\phi \quad \ldots\ldots (20b)$$

$$t_z = R_z \quad \ldots\ldots (20c)$$

$$m_u = R_x \varepsilon x \cos \phi + R_y \varepsilon x \sin \phi + R_z v' - N_x \sin \phi + N_y \cos \phi - W \varepsilon (x - x') \sin \phi + H \varepsilon (x - x') \cos \phi \quad \ldots\ldots (20d)$$

$$m_v = - R_x \varepsilon x \sin \phi + R_y \varepsilon x \cos \phi - R_z u' - N_x \cos \phi - N_y \sin \phi - W \varepsilon (x - x') \cos \phi - H \varepsilon (x - x') \sin \phi \quad \ldots\ldots (20e)$$

and,

$$m_z = R_x y + R_y x + N_z - W (x - x') + H (y - y') \quad \ldots\ldots (20f)$$

The equations of the co-ordinates of the center of the right abutment with respect to the point, P, of any section may be written as follows:

$$u' = x \sin \phi + y \cos \phi \quad \ldots\ldots (21a)$$

and,

$$v' = x \cos \phi - y \sin \phi \quad \ldots\ldots (21b)$$

The deformation formulas for this case are identical with Equation (7):

$$E\, d\, \gamma = m_z \frac{s}{I} \quad \ldots\ldots (22a)$$

and,

$$E\, d\, \alpha = m_v \frac{\lambda\, s}{F} \quad \ldots\ldots (22b)$$

Equations (20(e)) and (20f)) have been rewritten by taking each reaction or load in turn and making the remaining terms zero. Equations (23A), therefore, are for the stresses at a section due to each reaction component at the right abutment.

EQUATIONS (23A).—STRESSES AT A SECTION DUE TO REACTION COMPONENTS.

Reaction component.	m_z.	m_v.
R_x	$R_x y$	$-R_x \epsilon x \sin \phi$
R_y	$R_y x$	$R_y \epsilon x \cos \phi$
R_z	....	$-R_z u'$
N_x	....	$-N_x \cos \phi$
N_y	....	$-N_y \sin \phi$
N_z	N_z	.....

Stresses at a section due to a vertical load, W, at a point, $x = x'$, $z = \varepsilon x'$, are similarly given by:

$$\left.\begin{aligned} m_z &= - W (x - x') \\ \text{and,} \quad m_v &= - W \varepsilon (x - x') \cos \phi \end{aligned}\right\} \quad \ldots\ldots (23B)$$

Stresses at a section due to the horizontal load, H, at a point, $y = y'$, $z = \varepsilon x'$, are:

$$\left.\begin{aligned} m_z &= + H (y - y') \\ \text{and,} \quad m_v &= - H \varepsilon (x - x') \sin \phi \end{aligned}\right\} \quad \ldots\ldots (23C)$$

As in the case of the symmetrical arches, Equations (23A), when combined, are used, together with Equations (24B) and (24C).

Following the method used for the symmetrical arches, Equations (22) and (23) are substituted in Equation (24). For the fixed arch, all the deflections produced by the loads are equated to the deflections produced by the

reactions resulting in the final Equations (25), which are placed in the Appendix for the convenience of the designer. As in the case of Equations (16), the text numbering is used in the Appendix for convenience in referring back to the text when necessary.

EQUATIONS (24).—DIFFERENTIAL DEFLECTIONS OF THE RIGHT ABUTMENT.

Deflection.	Due to m_x.	Due to m_v.
$d\theta_x$	.....	$d\gamma \cos \phi$
$d\theta_y$	.....	$-d\alpha \sin \phi$
$d\theta_z$	$d\gamma$	.....
$d\Delta_x$	$d\gamma\ y$	$-d\alpha \epsilon x \sin \phi$
$d\Delta_y$	$d\gamma\ x$	$d\alpha \epsilon x \cos \phi$
$d\Delta_z$	.....	$d\alpha\ u'$

For the two-hinged arch, the fact that the left abutment is hinged prevents the assumption that the vertical deflections produced by the reactions and the loads are equal, and the fact that the right abutment is hinged prevents a similar assumption for the angular deflection of the right abutment about the Z-axis. Two additional equations are obtained from statics as in the case of the symmetrical arch, but in this case a correction must be made for the fact that the two abutments are not at the same elevation.

The structure has been transformed into a curved cantilever beam and the load has caused the right abutment to deflect vertically. Replacing the hinge in the left abutment, the entire arch is rotated until the right abutment returns to its original elevation. This rotation is about the Z-axis and corresponds to a γ-deflection. From Equation (24), it is seen that such an angular deflection produces a linear deflection along the X-axis, or a Δ_x. The Δ_x already derived for each load or reaction, therefore, must be corrected for this additional γ-deflection. These deflections will be denoted by the same symbols used previously, but primed.

In Fig. 10, the arch acting as a cantilever beam is deflected by the loads (not shown) so that B moves to B'. When the hinge at A is replaced, B' moves to B'', and the deflection, Δ_x, must be corrected by the amount, $B''\ D$.

Since γ' is small, $\angle\ AB''\ B'$ may be considered a right angle, so that $\angle\ B''\ B'\ D = \angle\ E$. From similar triangles: $\frac{a}{2\ L} = \frac{B''\ D}{B'\ D} = \frac{B''\ D}{\Delta_y}$ or,

$$B''\ D = \left(\frac{a}{2\ L}\right) \Delta_y$$

The final deflection along the X-axis is $B\ B''$; and $B\ B'' = B\ D - B''\ D$. Therefore,

$$\Delta'_x = \Delta_x - \left(\frac{a}{2\ L}\right) \Delta_y \qquad (26)$$

For the two-hinged unsymmetrical arch, the Δ'_x produced by the loads must equal the Δ'_x produced by the reactions. For the symmetrical arch, $a = 0$, and $\Delta'_x = \Delta_x$.

Four equations are obtained by equating the deflections, and two more are obtained from statics, resulting in the six final Equations (27). (See Appendix 1.)

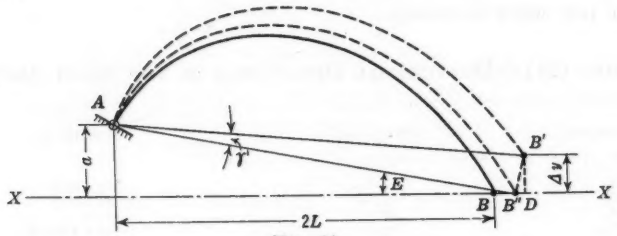


FIG. 10.

SUMMARY

This concludes Part I, which gives the derivation of the four sets of equations for obtaining the total stresses for the fixed and hinged arches, which may be either symmetrical or unsymmetrical. The unsymmetrical arch equations can be used for symmetrical arches, but would involve more numerical work. It should be remembered that the thrust, shears, and the bending deformations about the radial axes have been omitted. For large and important arches, the preceding theory should not be used without correction, or by an adjustment similar to that made for right arches.

PART II.—THEORY OF PROPORTIONING SECTIONS

The analysis of a structure may be divided into three parts:

(*a*) The computation of the forces and moments produced by the external loading at several sections of the structure.
(*b*) The distribution of the stresses over these sections.
(*c*) Design of the sections to resist the stresses acting on them.

The numerical work up to this point gives in detail all the steps necessary for plotting influence lines and the maximum moments, thrusts, and shears have been obtained from these curves in the usual way. The next step is to determine the distribution of the stresses over a section.

NOTATION

To the notation already given, the following symbols are added. In the design formulas, the standard reinforced concrete notation is used with the exceptions and additions noted. Symbols which are used only in the derivations, but which do not appear in the final formulas, are defined in the text and are not given here. Unless otherwise stated, symbols are always positive.

e = the distance from the intrados steel to the resultant normal thrust or horizontal shear acting on a section, and is positive in the direction away from the center of curvature of the arch.
e' = the same as e, except that it is measured from the extrados steel.
f_s = the unit stress in the steel in the intrados face; it is positive when the steel is in tension.
f_s' = the unit stress in the steel in the extrados face; it is positive when the steel is in tension.

f_c = the unit direct stress in the concrete; it is positive when the concrete is in compression.

h = the unit horizontal shearing stress in the concrete and is positive when it tends to rotate the concrete element on which it is acting, in the same direction in which a positive T_z (or t_z) tends to cause rotation.

j = the distance from the center of compression or the center of shear in the concrete to the intrados steel.

j' = the same as j, except that it is measured to the extrados steel.

q = a constant similar to k used in connection with the condition of compression or horizontal shear over the entire section. Its meaning is obvious from the text. It is positive when the stress at the intrados is numerically less than, or of opposite sign to, the stress at the extrados.

r = the distance between the intrados and extrados steel.

M = the direct moment per unit width acting on a design plane and is positive when it produces compression in the extrados fibers.

M_s = the horizontal shear moment per unit width acting on a design plane. Considering this moment as composed of two equal and opposite horizontal forces, it is positive when these forces tend to rotate the concrete element in the same direction in which the corresponding forces of the moment, m_v (or M_x), tend to cause rotation.

N = the direct thrust per unit width acting on a design plane and is positive when it produces compression.

S = the horizontal shear per unit width acting on a design plane and is positive in the same direction as h.

m = the direct moment per unit width acting on a plane perpendicular to the V-axis and is positive when it produces compression in the extrados fibers.

m' = the same as m, but acting on a plane perpendicular to the Z-axis.

t_n, t_n' = the direct thrust per unit width corresponding to m and m', respectively, and are positive when they produce compression.

m_s = the horizontal shear moment per unit width acting on a plane perpendicular to the V-axis and is positive in the same direction as m_v (or M_x).

m_s' = the horizontal shear moment per unit width acting on a plane perpendicular to the Z-axis and is positive in the same direction as M_s.

t_s = the horizontal shear per unit width corresponding to m_s and is positive in the same direction as t_z (or T_z).

t_s' = the horizontal shear per unit width corresponding to m_s' and is positive in the same direction as h.

V = the radial shear per unit width acting on a plane perpendicular to the V-axis and is positive in the same direction as t_u.

V' = the radial shear per unit width acting on a plane perpendicular to the Z-axis and is positive in the direction opposite to the positive direction of the shear, V, acting on the unbalanced triangle, $A\,C\,D$ (Fig. 16(a)).

α = the angle between the direction of the steel and the direction of a design plane, and is positive in the same direction as m_u (or M_y).

ψ = the angle between a plane perpendicular to the V-axis and a design plane and is positive in the same direction as m_u (or M_y).

θ' = the skew angle measured in the plane tangent to the neutral surface at any point. It corresponds to the angle, θ, measured in the horizontal plane.

ϕ' = the central angle measured in a plane parallel to the face of the bridge. It corresponds to the central angle, ϕ, measured in a plane perpendicular to the abutments.

Relation Between θ' and θ and Between ϕ' and ϕ

Before entering upon the second step of the analysis—the stress distribution—the relation between θ' and θ and between ϕ' and ϕ will be found.

From Fig. 11: $\tan \theta' = \cos \phi \tan \theta$, or,

$$\varepsilon' = \varepsilon \cos \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Also, $\tan \phi' = \dfrac{\sin \phi}{\cos \phi \sec \theta}$, or

$$\tan \phi' = \tan \phi \cos \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

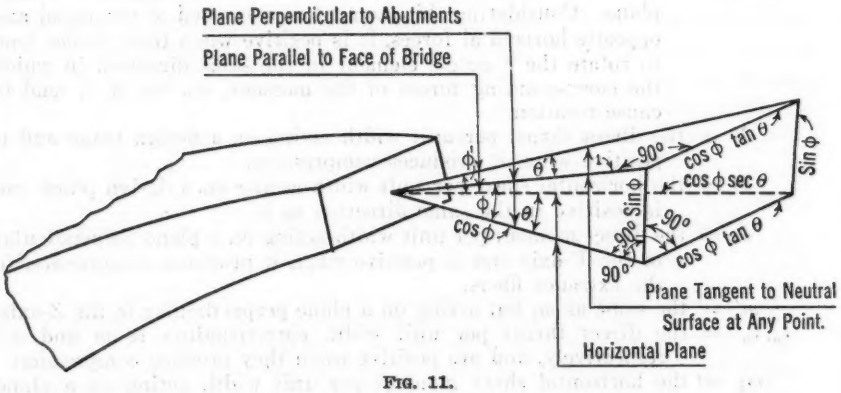


Fig. 11.

The relation between ϕ' and ϕ is convenient when laying out the arch for the purpose of scaling the properties of the sections. The intrados and extrados curves are usually determined in a plane parallel to the face of the bridge, while for the purposes of design, the arch must be laid out in a plane perpendicular to the abutments. This relation, therefore, gives a convenient method of finding ϕ which appears in the skew arch analysis. The relation between ϵ' and ϵ is important, because the former is used in the design of the steel at sections other than the crown.

Section 1.—Stress Distribution

Before the steel can be designed, the distribution of the stresses at any section produced by the various moments, shears, and thrusts must be determined. For simplicity, the crown section is analyzed in the introductory analysis, *A*, which follows. For other sections, the analysis is the same except that the sections are not vertical and the planes tangent to the neutral surface are not horizontal. The angle, θ', is then used instead of the angle, θ.

(*A*) Stresses at Face of Bridge

Resultant Thrust at Face of Bridge.—In Fig. 12(*a*), f is the unit stress produced at A by the direct thrust, T_x. Since there is no material to the left of the section to resist the stress, it is impossible for f to act by itself. There must be, therefore, a unit shear acting at the point, A, of such a magnitude that the resultant stress will be parallel to the face of bridge as shown in Fig. 12(*b*). The resultant stress is evidently equal to f sec θ, and the unit shear required to bring this stress in line with the face of bridge is εf.

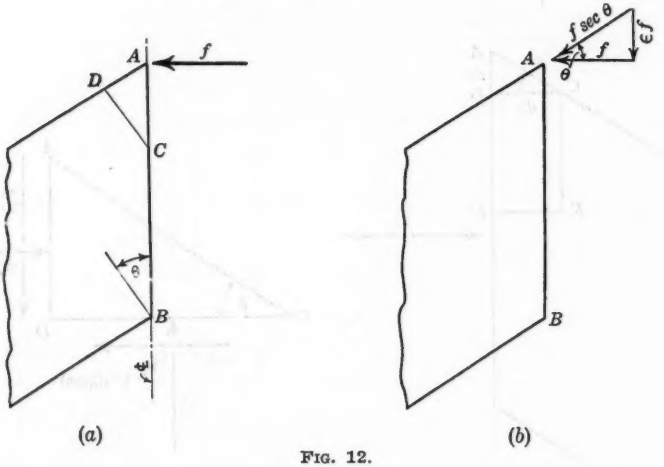


FIG. 12.

Another way of arriving at the same result is as follows: In Fig. 12, $C\,D$ is a plane perpendicular to the face of the bridge. The stress, f, produces a unit shear on the plane, $C\,D$, and an equal unit shear on the plane, $A\,D$, which is perpendicular to the plane, $C\,D$. However, the plane, $A\,D$, is an external face and can carry no stress. Hence, there can be no shear on Plane $C\,D$, and the resultant unit stress must be normal to it, that is, parallel to the face of bridge.

Thrust and Shears on Plane Perpendicular to Z-Axis.—In Fig. 13, a thrust, T_x, acts on the crown section, $A\,B$. At A, there is a direct unit stress, f, and a unit shear, εf, as previously shown. If the triangle, $A\,C\,D$, of dimensions, dx, dz, is cut from the arch, this triangle shown enlarged in Fig. 14, is a free body and must be in equilibrium. Face $A\,C$ is an external face in the uncut arch and carries no stress. For equilibrium there must be a unit shear, h, and unit thrust, t_n, on the face, $C\,D$.

Since $\Sigma X = 0$, $-h\,dx = f\,dz$; $-h = f\,\dfrac{dz}{dx} = \varepsilon f$; and,

$$t'_s = -t_s = -\varepsilon f \qquad (30)$$

This merely proves the familiar theorem that if there is a unit shear on one plane, there must be an equal unit shear on a plane at right angles to it.

Again, for equilibrium since $\Sigma Z = 0$, $t'_n \, dx = \varepsilon f \, dz$; $t'_n = \varepsilon f \dfrac{dz}{dx} = \varepsilon^2 f$; and,

$$t'_n = \varepsilon^2 t_n = \varepsilon t_s \quad \ldots\ldots (31)$$

Furthermore, since $\Sigma Y = 0$, $V' \, dx = \dfrac{T_y}{b} dz = V \, dz$, and therefore,

$$V' = \varepsilon V \quad \ldots\ldots (32)$$

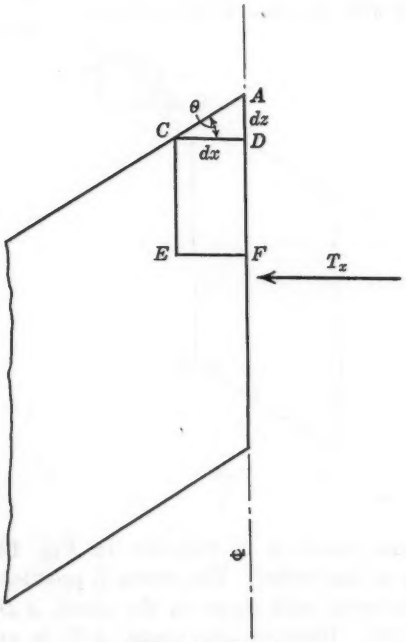


FIG. 13.—PLAN.

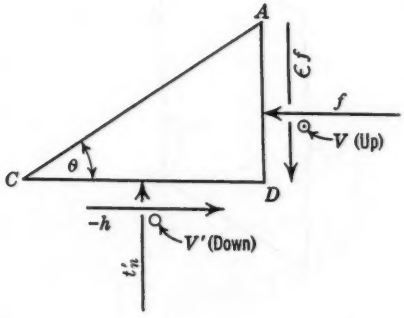


FIG. 14.

Direct Moment and Shear Moment on Plane Perpendicular to Z-Axis.—In the discussion of the direct thrust, it was shown that at the face of bridge a shear is present which makes the resultant thrust act parallel to that face. It was also shown that there is a direct thrust acting on the plane perpendicular to the section. Similar reasoning holds true for the stresses produced by M_z; but, whereas the stresses produced by the thrust, T_x, are uniformly distributed over the depth of the section, the distribution of M_z is variable.

In Fig. 15(*a*) a bending moment, M_z, acts on the crown section, AB. At A, f is the extreme fiber stress produced by the bending moment, and $h = \varepsilon f$ is the corresponding unit shear necessary to make the resultant stress act parallel to the face of bridge. The fiber stress at any other point on the radial line through A is f' (Fig. 15(*b*)). A corresponding unit shear, $h' = \varepsilon f'$, must accompany this fiber stress, f', in order that the resultant stress shall here also be parallel to the face of bridge; but f' varies along the depth, t, of the sec-

tion. Therefore, h' must also vary along the depth of the section. The relation between these variables can be expressed in terms of moments. That is, $M_z = b \Sigma f' y \, dA$, or, $m = \frac{M_z}{b} = \Sigma f' y \, dA$; also, $\Sigma \varepsilon f' y \, dA = \Sigma h' y \, dA$. Therefore,

$$\varepsilon m = m_s \quad \text{.....(33)}$$

and, similarly,

$$\varepsilon^2 m = m' \quad \text{.....(34)}$$

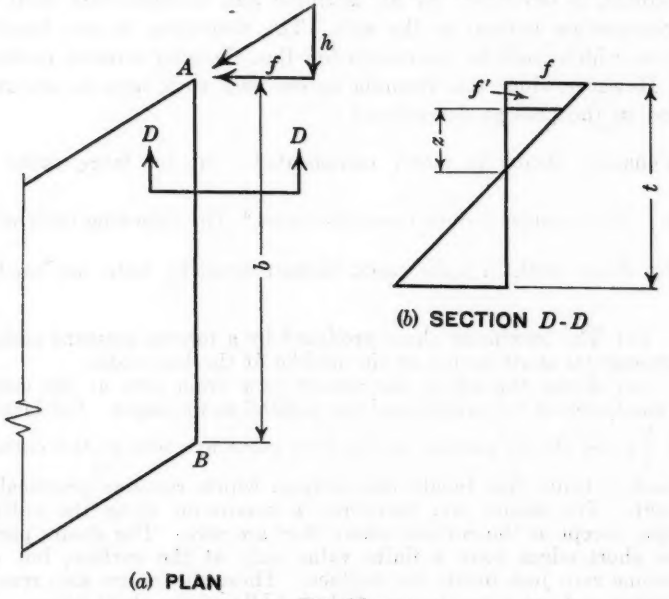


FIG. 15.

The moment, m_s, is part of a torsion moment. It is the summation of the moments of the horizontal shears about the horizontal center of gravity of the section. It is also the shear moment required to make the resultant bending stresses at the face of the bridge act parallel to that face.

(*B*) Stresses at Any Point of a Section

Equations (30) to (34), inclusive, give the value of certain stresses at the face of bridge. The values of these stresses at any point of the section will now be found. As in the right arch, the effect of the curvature of the arch on the stress distribution will be neglected. The error involved is small for curves of large radii.

In the following discussion, the problem of a shaft subjected to torsion is encountered again. Before a correct design of a skew arch can be made, a correct torsion formula will have to be developed. Of course, the method presented herein is not exact, but it is developed with the idea of obtaining a logical design with the data available and of pointing out some of the problems involved.

There are various formulas for analyzing a shaft subjected to torsion. Most of them are very approximate except for the circular and the square sections. The usual assumptions are that an unstressed plane section remains plane after it is stressed and that only shearing stresses are produced. These assumptions are not true except in some special cases. St. Venant developed a more accurate theory which shows that a plane surface becomes warped when the shaft is twisted. This theory, like the more approximate theories, is developed for an isotropic and homogeneous shaft of constant cross-section normal to the axis. The derivation is also based on an assumption which leads to the conclusion that the only stresses produced are shears. However, since this formula agrees with such tests as are available, it is used in the present derivation.[3]

St. Venant's theory is rather complicated; but for large ratios of $\frac{b}{t}$, $\left(\frac{b}{t} > 3\right)$, the formulas reduce to simple terms.[4] The following facts which are proved by theory and, in some cases, substantiated by tests, are used in the design:

(*a*) The maximum shear produced by a torsion moment acting on a rectangular shaft occurs at the middle of the long side.

(*b*) Along the edges, the shears vary from zero at the corners to a maximum at the middle and are parallel to the edges. For large ratios of $\frac{b}{t}$, the shears parallel to the long edges are zero at the corners, but reach a value just inside the surface, which remains practically constant. The shears are, therefore, a maximum along the entire long edge, except at the corners where they are zero. The shears parallel to the short edges have a finite value only at the surface, but quickly become zero just inside the surface. These shears are also zero at the corners and are a maximum at the middle of the short edges.

(*c*) The torsion moment can be divided into two parts, one-half of which produces shears parallel to the long edge and the other half, shears parallel to the short edge.

In the following derivation, it is assumed that the thrust, (T_x, t_v), the bending moment, (M_z, m_z), and the radial shears, (T_y, t_u), are uniformly distributed over the width, b, of any radial section parallel to the center line of the span, (Z-axis).

Distribution of Horizontal Shear.—Fig. 16(*a*) is a plan view of a skew frame showing the position of a strip of differential width. The strip is enlarged in Fig. 16(*b*) for convenience in illustrating the special notation that applies. In these diagrams, let,

t_3 = unit radial shear produced by the net torsion moment;

Q = static moment of the area, $A\ F$, about the axis, $U - U$

$$= \frac{1}{2}\left(\frac{b^2}{4} - z^2\right) t;$$

[3] A short discussion of torsion is given in *Bulletin No. 3* of the University of Toronto, previously mentioned.

[4] "Applied Elasticity," by John Prescott.

$A = b\,t$;

$a = A\,F \times t = \left(\frac{b}{2} - z\right) t$; and,

$dv = dx$ (neglecting the curvature of the arch).

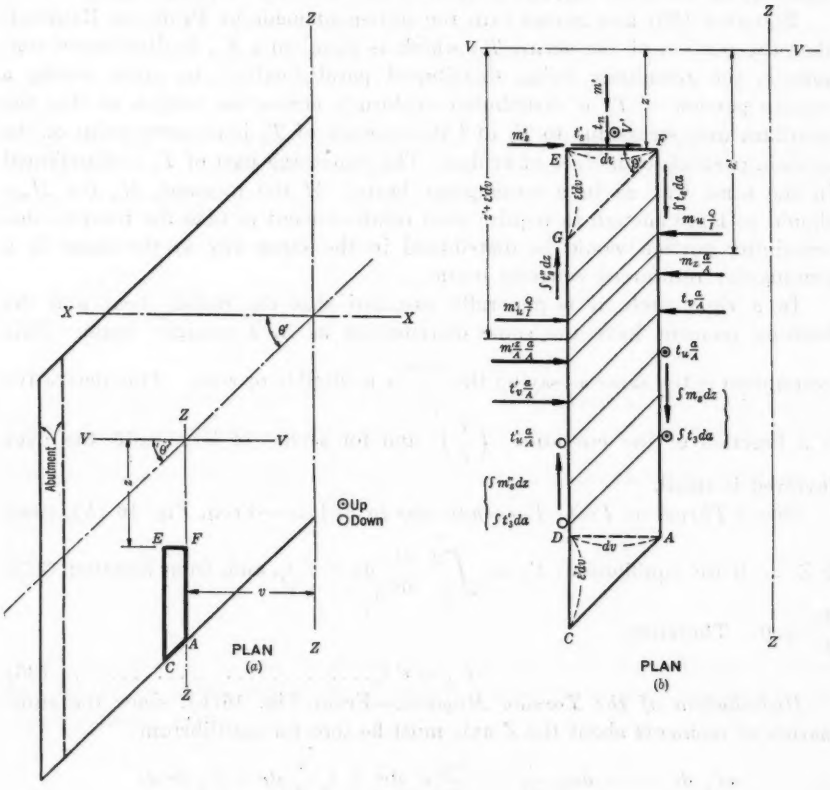


FIG. 16.

Moments are represented as two equal and opposite forces of which only the upper force is shown in the diagrams.

Since $\Sigma\ V = 0$ for equilibrium: $t'_s\ dv = -\ m'_u\ \frac{Q}{I} + m_u \frac{Q}{I} - m_u \frac{z}{I}\ \varepsilon'\ dv$ $- t'_v \frac{a}{A} + t_v \frac{a}{A} - \frac{t'_v}{b}\ \varepsilon'\ dv$, or, $t'_s = -\frac{dm_u}{dv}\frac{Q}{I} - \varepsilon'\ m_u \frac{z}{I} - \frac{dt_v}{dv}\frac{a}{A} - \varepsilon'\ \frac{t_v}{b}$.

From Equations (2), in Part I, neglecting the effect of the curvature of the arch; that is, making $\frac{d\phi}{dv} = 0$; $\frac{dm_u}{dv} = (t_z - \varepsilon'\ t_v)$; $\frac{dt_v}{dv} = 0$. Furthermore, $t_s = -\ t'_s$, and therefore,

$$t_s = \frac{3}{2}\left(\frac{t_z - \varepsilon'\ t_v}{b}\right)\left(1 - 4\,\frac{z^2}{b^2}\right) + \varepsilon'\ \frac{t_v}{b} + 12\ \varepsilon'\ \frac{m_u\ z}{b^3} \quad \ldots\ldots\ldots (35)$$

The last term of Equation (35) is usually small and can be neglected.

This equation shows that the shear, t_s, is composed of two parts: (*a*), that produced by the computed shear, t_z; and (*b*), that produced by the effect of the skew on the thrust, t_v. When $\epsilon' = 0$, the shear, t_z, has the usual parabolic distribution for a rectangular section.

Equation (35) also agrees with the statement made by Professor Rathbun[5] that the portion of the shear, T_z, which is equal to $\epsilon\, T_x$, is distributed uniformly, the remainder being distributed parabolically. In other words, a certain portion of T_z is distributed uniformly across the section so that the resultant unit stress due to T_x and this portion of T_z is at every point of the section, parallel to the face of bridge. The remaining part of T_z is distributed in the same way as in a rectangular beam. If the moment, M_y (or M_u), should be large enough to require steel reinforcement to take the tension, this remaining portion would be distributed in the same way as the shear in a rectangular reinforced concrete beam.

In a right arch, it is generally assumed that the radial shear and the bending moment have the same distribution as in a straight beam. This assumption is the same as saying that $\frac{d\phi}{dv}$ is negligible or zero. This derivative is a function of the curvature, $\left(\frac{1}{r}\right)$, and for arches of large radii, the error involved is small.

Direct Thrust on Plane Perpendicular to Z-Axis.—From Fig. 16 (*b*), since $\Sigma Z = 0$ for equilibrium: $t'_n = \int_z^{\frac{b}{2}} \frac{\delta t_s}{\delta v}\, dz + \varepsilon'\, t_s$, and, from Equation (35): $\frac{\delta t_s}{\delta v} = 0$. Therefore,

$$t'_n = \varepsilon'\, t_s \qquad (36)$$

Distribution of the Torsion Moment.—From Fig. 16(*b*), since the summation of moments about the *Z*-axis must be zero for equilibrium:

$$m'_s\, dv = -\, dm_z\, \frac{a}{A} - \frac{m_z}{b}\, \varepsilon'\, dv + t_u \frac{a}{A}\, dv + t'_u\, dv\, dz$$

$$+ \left(\int_z^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\, dz\, du \right) dv + \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\, du \right) dv\, dz.$$

Furthermore, since $m_s = -\, m'_s$ and $\frac{\delta m_z}{\delta v} = t_u$,

$$m_s = \frac{\varepsilon'\, m_z}{b} - \int_z^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\, dz\, du \qquad (37)$$

Equation (37) is similar to Equation (35) which shows that a certain portion of the horizontal shear, t_z, is distributed uniformly and the remainder is distributed the same as a shear in a similar rectangular (unskewed) beam. Similarly, Equation (37) shows that a portion of the torsion moment equal

[5] *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 678.

to $\varepsilon' m_z$ is distributed uniformly so that the resultant unit stresses produced by this moment and the direct moment, m_z, are at each point of the section parallel to the face of the bridge, and the remaining portion is distributed like a torsion moment. The two portions of the torsion moment are given by the first and second terms, respectively, of the right-hand member of Equation (37). The amount of the portion of this moment which is distributed like a torsion moment is still to be found. This unknown moment will be denoted by m'_v.

In dealing with the torsion moment, m'_v, St. Venant's torsion formulas will be used.[6] A correct design is not possible until the problem of torsion in general is studied further and tests are made.

Integrating both sides of Equation (37) between the limits of $\frac{b}{2}$ and $-\frac{b}{2}$:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} m_s\, dz = \varepsilon' m_z - \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{z}^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\, dz\, dz\, du \quad \ldots\ldots\ldots (38)$$

Substituting the value of t_3 in terms of m_v', as given by St. Venant's formulas, and performing the integrations indicated, the last term of Equation (38) can be shown to equal $-\frac{1}{2} m'_v$. Equation (38) can then be rewritten as follows:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} m_s\, dz - \frac{1}{2} m'_v = \varepsilon' m_z \quad \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Since the sum of the moments produced by both the horizontal shears and the radial shears acting on the entire cross-section must equal the computed moment, m_v:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} m_s\, dz - \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\, z\, dz\, du = m_v \quad \ldots\ldots\ldots\ldots (40)$$

St. Venant's formulas show that the second term of Equation (40) also equals $-\frac{1}{2} m'_v$. Therefore,

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} m_s\, dz + \frac{1}{2} m'_v = m_v \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

Eliminating the term, $\int_{-\frac{b}{2}}^{\frac{b}{2}} m_s\, dz$, between Equations (39) and (41), and solving for m'_v:

$$m'_v = m_v - \varepsilon' m_z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

[6] These formulas are developed by Prescott in his "Applied Elasticity," Chapter VII, pp. 148 to 156, and are used because they are the best available.

The equation for m_s can now be found. At the face of the bridge, the second term of Equation (37) is given by St. Venant's formulas as zero. At any other point, this second term reduces to a value which for large ratios of $\frac{b}{t}$ $\left(\text{that is, } \frac{b}{t} > 3\right)$, is nearly constant and equals $-\frac{m'_v}{2\,b}$. Equation (37), therefore, reduces to the following expression:

At the face of the bridge:

$$m_s = \frac{\varepsilon'\, m_z}{b} \qquad (43)$$

At any other point of the section:

$$m_s = \frac{m_v + \varepsilon'\, m_z}{2\,b} \qquad (44)$$

Two equations for the shear moment per unit width of section are necessary because, for large ratios of $\frac{b}{t}$, the torsion moment acting on a section which is normal to the sides of the shaft, produces stresses which are either zero at the surface of the short side and quickly reach a finite value just inside the surface; or, they have a finite value at the surface, and quickly reach zero just inside the surface. In a skew section, the surface stresses do not reach zero, but become a minimum which is a function of the skew angle and is produced by the effect of the skew on the bending moment, m_z.

Attention is called to the fact that the previous sentence is really the crux of the problem of the stress distribution on a skew section. The fact that the stresses on the section at the surface must be parallel to that surface is what causes the difference between the distribution on a right section and a skew section.

Bending Moment on Plane Perpendicular to Z-Axis.—From Fig. 16(*b*), since the summation of moments about the V-axis must be zero for equilibrium:

$$m'\,dv = \int_z^{\frac{b}{2}} m''_s\,dz - \int_z^{\frac{b}{2}} m_s\,dz + m_s\,\varepsilon'\,dv - \int_z^{\frac{b}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}} t'_3\,(z' + \varepsilon'\,dv - z)\,dz\,du$$

$$+ \int_z^{\frac{b}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\,(z' - z)\,dz\,du - \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\,du\right)(\varepsilon'\,dv)\left(\varepsilon'\,\frac{dv}{2}\right)$$

$$- t'_u\,\frac{a}{A}\left[\frac{1}{2}\left(\frac{b}{2} - z\right) + \varepsilon'\,dv\right] + t_u\,\frac{a}{A}\left[\frac{1}{2}\left(\frac{b}{2} - z\right)\right] - t'_u\,(\varepsilon'\,dv)\left(\varepsilon'\,\frac{dv}{2}\right)$$

Combining and reducing,

$$m' = \int_z^{\frac{b}{2}} \frac{\delta\, m_z}{\delta\, v}\,dz + \varepsilon'\,m_s - \int_z^{\frac{b}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{\delta\, t_3}{\delta\, v}\,(z' - z)\,dz'\,du$$

$$- \varepsilon'\int_z^{\frac{b}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}} t_3\,dz\,du - \frac{1}{2}\,\frac{\delta\, t_u}{\delta\, v}\,\frac{a}{A}\left(\frac{b}{2} - z\right) - \varepsilon'\,t_u\,\frac{a}{A} \qquad (45)$$

From St. Venant's formulas: $t_3 = -m'_v f(z) = -(m_v - \varepsilon' m_z) f(z)$. Neglecting the curvature of the arch: $\frac{\delta t_3}{\delta v} = 0$; $\frac{\delta t_u}{\delta v} = 0$. Differentiating Equation (37): $\frac{\delta m_s}{\delta v} = \frac{\varepsilon' t_u}{b}$.

Substituting these values in Equation (45):

$$m' = \varepsilon' m_s - \varepsilon' \int_z^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} t_3 \, dz \, du \quad \ldots\ldots (46)$$

Eliminating the double integral between Equations (37) and (46),

$$m' = 2 \varepsilon' m_s - \varepsilon'^2 \frac{m_z}{b} \quad \ldots\ldots (47)$$

Referring to Equations (43) and (44), Equation (47) reduces to the following expressions:

At the face of bridge:

$$m' = \frac{\varepsilon'^2 m_z}{b} \quad \ldots\ldots (48)$$

At any other point of the section:

$$m' = \frac{\varepsilon' m_v}{b} \quad \ldots\ldots (49)$$

Equation (49), like Equation (44) holds only for large ratios of $\frac{b}{t}$ $\left(\text{that is, } \frac{b}{t} > 3\right)$, which is usually the case for arches.

Radial Shear on Plane Perpendicular to Z-Axis.—From Fig. 16(*b*), since $\Sigma U = 0$ for equilibrium:

$$V' = \frac{\delta t_u}{\delta v}\left(\frac{a}{A}\right) + \varepsilon' \frac{t_u}{b} + \int_z^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{\delta t_3}{\delta v} \, dz \, du + \varepsilon' \int_{-\frac{t}{2}}^{\frac{t}{2}} t_3 \, du \quad \ldots (50)$$

At the face of bridge:

$$V' = \varepsilon' \frac{t_u}{b} - \frac{8}{5} \varepsilon' \frac{(m_v - \varepsilon' m_z)}{b\,t} \quad \ldots\ldots (51)$$

At any other point of the section:

$$V' = \varepsilon' \frac{t_u}{b} \quad \ldots\ldots (52)$$

Section 2.—Maximum Stresses and Transformation Formulas

In the design of a right rigid frame in which the fiber stresses produced by the moments are large compared to those produced by the direct thrusts, the live load thrusts are found with the live load in the position to produce maximum moments. The error involved is small. The same condition holds true in a skew arch, except that there are two moments, a thrust, and a shear: M_x, M_z, T_x, and T_z (or m_v, m_z, t_v, and t_z), that must be computed for the

same position of live load instead of the single moment and thrust in the right arch and the position of the live load is governed by M_z. The problem of vertical shear or, more accurately, the radial shear, is treated in the same manner as in the right arch and needs no special consideration.

The maximum stresses for any given condition of loading occur on the principal planes. Steel can be designed in any direction to take care of the principal stresses, but if it were placed in one direction only, it would not resist the tension on a plane parallel to itself. Hence, the steel must be placed in two directions. However, while both sets of steel resist the tension on the principal planes, only one set acts on the plane parallel to the other. Therefore, more steel may be required on a plane parallel to one set of steel than on the principal planes. Reinforcement should be designed on the planes parallel to the direction of the steel, and also on the principal planes. Furthermore, the maximum horizontal shear will occur on still another plane. In order to analyze the stresses on these various planes, the stresses on a section must be transformed to give other stresses which are parallel and perpendicular to the planes in question.

The required transformation formulas can be obtained from any textbook on mechanics. They are given without proof in Appendix 1 in terms of the notation already adopted.

Section 3.—Effect of Angle Between Direction of Steel and Direction of Stress

It was shown in Section 2, that the direction of the principal stresses (which are the maximum and the minimum), with respect to the character of the center line of the arch, are determined by the relative magnitudes of the forces acting on a section. While the direction of the stress varies, it is not practical to vary the direction of the steel. The angle made by the steel with the direction of the design stresses introduces certain factors which must be taken into account.

Before going further, it is well to point out the following more or less obvious fact. A steel reinforcing rod can take practically no bending. In tension, therefore, it acts like a string. If the stresses at a point are analyzed on different planes, the stress in the concrete will vary, but that in the steel at any point must remain constant. In other words, at a given point, the value, $f_s\ A_s$, is constant (for the same condition of loading) no matter on what plane the stresses are analyzed.

(*A*) Relation of Stress in Steel to Components of Direct Stress Deformation

In Fig. 17, $A\ D\ B\ F$ is an element of concrete with a strand of reinforcing steel, $A\ B$, embedded in it. Forces producing tension and compression are acting on the four faces, $A\ D$, $B\ F$, $B\ D$, and $A\ F$, of the element so that it deflects in both directions. Point A is taken as the origin so that $B\ H$ and $B\ G$ are deflections for the lengths, $A\ F$ and $B\ F$. Point B moves to B', and the new length of the strand is now $A\ B'$.

From Fig. 17, since Angle $B\ A\ M$ is small and $A\ B = A\ M$, Angle $B\ M\ A$ may be considered a right angle. Let δ_1 be the unit elongation in the direction, $A\ F$; δ_2, the unit elongation in the direction, $B\ F$; and, f_d, the stress in the steel due to the direct stresses in the concrete. Then,

$$\delta_1 = \frac{B\ H}{B\ D} = \frac{B\ B' \cos \beta}{A\ B \cos \alpha} = \frac{B'\ M \sec (\alpha - \beta) \cos \beta}{A\ B \cos A}$$

or,

$$\delta_1 = \frac{f_d}{E_s} \frac{\cos \beta}{\cos \alpha \cos (\alpha - \beta)} \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

Also,

$$\delta_2 = \frac{B\ G}{B\ F} = \frac{B\ B' \sin \beta}{A\ B \sin \alpha} = \frac{B'\ M \sin \beta}{A\ B \sin \alpha \cos (\alpha - \beta)}$$

or,

$$\delta_2 = \frac{f_d}{E_s} \frac{\sin \beta}{\sin \alpha \cos (\alpha - \beta)} \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

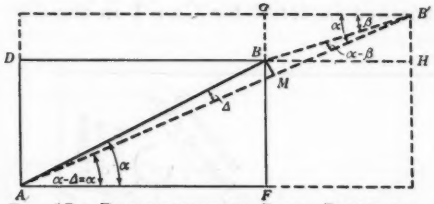


FIG. 17.—DEFORMATION IN STEEL PRODUCED BY DIRECT DEFORMATION.

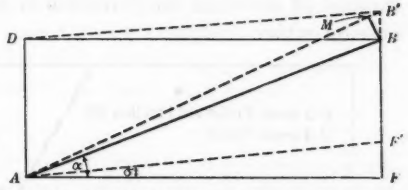


FIG. 18.—DEFORMATION IN STEEL PRODUCED BY SHEAR DEFORMATION IN CONCRETE.

Dividing Equation (54) by Equation (55), $\frac{\delta_1}{\delta_2} = \frac{\tan \alpha}{\tan \beta}$, or,

$$\tan \beta = \frac{\delta_2}{\delta_1} \tan \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

From Equation (54)

$$\delta_1 = \frac{f_d}{E_s \cos \alpha (\cos \alpha + \sin \alpha \tan \beta)} \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

Substituting Equation (56) in Equation (57), and transforming:

$$\frac{f_d}{E_s} = \delta_1 \cos^2 \alpha + \delta_2 \sin^2 \alpha \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

(*B*) Relation of Stresses in Steel to Shear Deformation

From Fig. 18, neglecting the squares of small quantities: $(A\ M + B'\ M)^2 = (A\ F)^2 + (B\ F + B\ B')^2$, or $F\ F' = \frac{B'\ M \times A\ M}{B\ F}$. Writing for $F\ F'$ its value, $A\ F \times \sigma$, and assuming that f_v equals the stress in the steel due to the shear in the concrete; then, $\sigma = \frac{B'\ M}{A\ F} \times \frac{A\ M}{B\ F} = \frac{B'\ M}{A\ B \cos \alpha} \times \frac{1}{\sin \alpha}$. Therefore,

$$\frac{f_v}{E_s} = \sigma \sin \alpha \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59)$$

Combining Equations (58) and (59) to get the total stress in the steel,

$$\frac{f_s}{E_s} = \delta_1 \cos^2 \alpha + \delta_2 \sin^2 \alpha + \sigma \sin \alpha \cos \alpha \ldots\ldots\ldots\ldots (60)$$

Equation (60) gives the relation between the stress in the steel, the components of the elongation of the steel in the direction of the direct stresses on the faces of the concrete element, and the shearing deformation of that element.

In the usual simple case where the steel is parallel to the direction of the applied stress, the deformations in the steel and in the concrete are proportional to their distances from the neutral axis. In the present case, the components of the deformations in the steel and in the concrete in the same direction are proportional to their distances from their respective neutral axes.

Section 4.—Derivation of Design Formulas

The design formulas can be divided into twelve cases, each one of which has three sub-divisions. Each of these sets of formulas can be tabulated. The method of deriving the formulas is explained in detail for Case I in the discussion which follows.

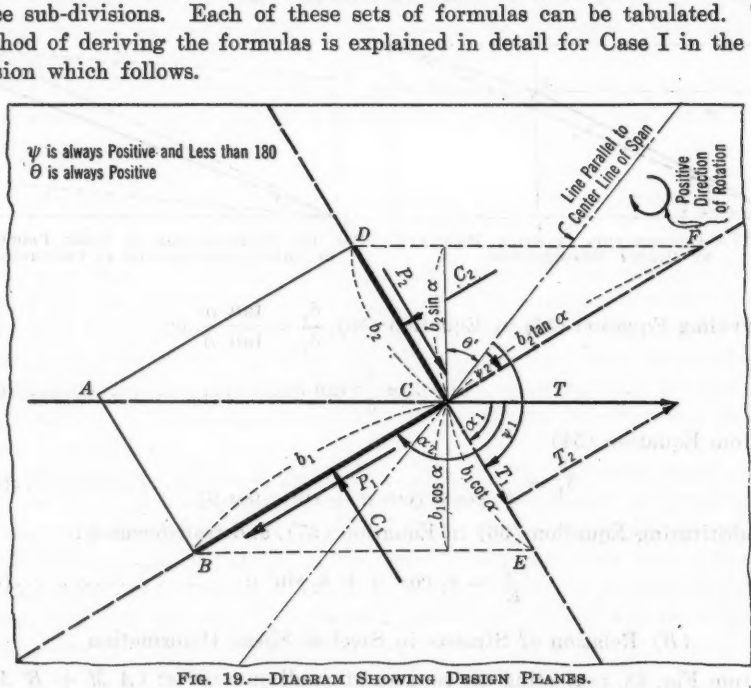


Fig. 19.—Diagram Showing Design Planes.

Case I.—Simple Bending—Steel in One Direction and in One Face Only

In Fig. 19, *A B C D* is an element of concrete of small dimensions in which a strand of steel is embedded. The planes, *B C F* and *D C E*, are any two planes at right angles and will be referred to as "Design Planes". The subscript numeral, 1, is attached to all symbols on the plane, *B C F*, and the subscript numeral, 2, to all symbols on the plane, *D C E*.

The values of α for various conditions are:

Longitudinal steel, parallel to fascia.....	$(\psi + \theta') - 90°$
Longitudinal steel, perpendicular to abutment..	$\psi - 90°$
Cross-steel, perpendicular to fascia.....	$\psi + \theta'$
Cross-steel, parallel to abutments.....	ψ

In the following derivations, Poisson's ratio is neglected because the value of this modulus is about the same for both concrete and steel. Poisson's ratio affects only the relative deformations and, since the value is about the same for both materials, there is no differential action on this account, and, therefore, no change in the relative stresses is produced. The stress deformation diagrams for each design plane are drawn as if there were no stress on the plane at right angles to it. The deformations shown are, therefore, only those produced by the stresses acting on the plane in question.

(*A*).—*Tension Over Part of Section on Both Design Planes and on the Same Side of the Section.*—For this case, assuming the usual straight-line distribution of flexural stress, the stress deformation diagrams on Planes *B C F* and *D C E*, Fig. 19, are shown in Fig. 20(*b*) and Fig. 20(*c*). At the middle of the cross-section, the torsion moment stresses also have a straight-line distribution, and the resulting stress deformation diagram, taking into account the shear produced by the angularity of the steel, is as shown in Fig. 20(*d*).

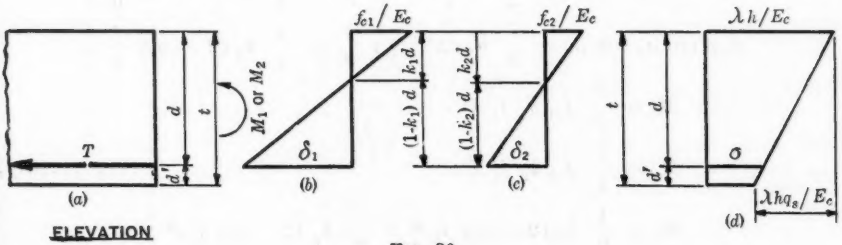


FIG. 20.

From Figs. 19 and 20:

$$\left.\begin{aligned} T &= f_s A_s (b_1 \cos \alpha_1 + b_2 \sin \alpha_1) \\ T_1 &= T \cos \alpha_1 = \begin{cases} \text{tension over } B\,F \\ \text{shear over } D\,E \end{cases} \\ T_2 &= T \sin \alpha_1 = \begin{cases} \text{tension over } D\,E \\ \text{shear over } B\,F \end{cases} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (61a)$$

$$\left.\begin{aligned} C_1 &= \frac{1}{2} f_{c1}\, k_1\, b_1\, d \\ C_2 &= \frac{1}{2} f_{c2}\, k_2\, b_2\, d \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (61b)$$

$$\left.\begin{aligned} P_1 &= \frac{1}{2} h_1 (2 - q)\, b_1\, t \\ P_2 &= \frac{1}{2} h_2 (2 - q)\, b_2\, t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (61c)$$

On Plane $B\,C$, since the summation of all forces parallel and perpendicular to the plane must be zero for equilibrium:

$$\left.\begin{aligned} \frac{C_1}{b_1} &= \frac{T_1}{b_1 + b_2 \tan \alpha_1} \\ \frac{P_1}{b_1} &= \frac{T_2}{b_1 + b_2 \tan \alpha_1} \end{aligned}\right\} \quad \ldots\ldots (61d)$$

Similarly, on Plane $C\,D$:

$$\left.\begin{aligned} \frac{C_2}{b_2} &= \frac{T_2}{b_1 \cot \alpha_1 + b_2} \\ \frac{P_2}{b_2} &= \frac{T_2}{b_1 \cot \alpha_1 + b_2} \end{aligned}\right\} \quad \ldots\ldots (61e)$$

Substituting the values from Equations (61*a*), (61*b*), and (61*c*) in Equations (61*d*) and (61*e*), and writing $p\,d$ for A_s:

$$\left.\begin{aligned} f_s\,p \cos^2 \alpha_1 &= \frac{1}{2}\, f_{c1}\, k_1 \\ f_s\,p \sin^2 \alpha_1 &= \frac{1}{2}\, f_{c2}\, k_2 = f_s\,p \cos^2 \alpha_2 \\ f_s\,p \sin \alpha_1 \cos \alpha_1 &= \frac{1}{2}\, h_1\,(2 - q_s)\,\frac{t}{d} = \frac{1}{2}\, h_2\,(2 - q_s)\,\frac{t}{d} \\ f_s\,p \sin \alpha_2 \cos \alpha_2 &= \frac{1}{2}\, h_1\,(2 - q_s)\,\frac{t}{d} = \frac{1}{2}\, h_2\,(2 - q_s)\,\frac{t}{d} \end{aligned}\right\} \quad \ldots (61f)$$

$$\left.\begin{aligned} M_1 &= \frac{1}{2}\, f_{c1}\, k_1\, j_1\, d^2 \\ M_2 &= \frac{1}{2}\, f_{c2}\, k_2\, j_2\, d^2 \\ M_s &= \frac{1}{2}\, h_1\,(2 - q_s)\, j_s\, t^2 = \frac{1}{2}\, h_2\,(2 - q_s)\, j_s\, t^2 \end{aligned}\right\} \quad \ldots\ldots (61g)$$

$$\left.\begin{aligned} M_1 &= f_s\,p\, j_1\, d^2 \cos^2 \alpha_1 \\ M_2 &= f_s\,p\, j_2\, d^2 \sin^2 \alpha_1 = f_s\,p\, j_2\, d^2 \cos^2 \alpha_2 \\ M_s &= f_s\,p\, j_s\, dt \sin \alpha_1 \cos \alpha_1 = f_s\,p\, j_s\, d\,t \sin \alpha_2 \cos \alpha_2 \end{aligned}\right\} \quad \ldots\ldots (61h)$$

$$\left.\begin{aligned} j_1 &= 1 - \frac{k_1}{3} \\ j_2 &= 1 - \frac{k_2}{3} \\ j_s &= \frac{d}{t} - \frac{3 - 2\,q_s}{6 - 3\,q_s} \end{aligned}\right\}. \quad \ldots\ldots (61i)$$

From Fig. 20:

$$\left.\begin{aligned} \delta_1 &= \frac{f_{c1}}{E_c}\left(\frac{1 - k_1}{k_1}\right) \\ \delta_2 &= \frac{f_{c2}}{E_c}\left(\frac{1 - k_2}{k_2}\right) \\ \sigma &= -\frac{\lambda\, h_1}{E_c}\left(1 - q\,\frac{d}{t}\right) \end{aligned}\right\} \quad \ldots\ldots (61j)$$

Substituting the values from Equations (61*j*) in Equation (60):

$$\left.\begin{aligned}\frac{f_s}{n} &= f_{c1}\left(\frac{1-k_1}{k_1}\right)\cos^2\alpha_1 + f_{c2}\left(\frac{1-k_2}{k_2}\right)\sin^2\alpha_1 - \lambda\, h_1\left(1-q_s\frac{d}{t}\right)\sin\alpha_1\cos\alpha_1\\ \frac{f_s}{n} &= f_{c1}\left(\frac{1-k_1}{k_1}\right)\sin^2\alpha_2 + f_{c2}\left(\frac{1-k_2}{k_2}\right)\cos^2\alpha_2 - \lambda\, h_2\left(1-q_s\frac{d}{t}\right)\sin\alpha_2\cos\alpha_2\end{aligned}\right\} \quad (61k)$$

Eliminating f_s, f_{c1}, and f_{c2} between Equations (61*f*) and Equations (61*k*):

$$\left.\begin{aligned}\frac{1}{2\,p\,n} &= \left(\frac{1-k_1}{k_1^2}\right)\cos^4\alpha_1 + \left(\frac{1-k_2}{k_2^2}\right)\sin^4\alpha_1\\ &\quad - \lambda\left(\frac{d}{t}\right)\left(\frac{1-q_s\dfrac{d}{t}}{2-q_s}\right)\sin^2\alpha_1\cos^2\alpha_1\\ \frac{1}{2\,p\,n} &= \left(\frac{1-k_1}{k_1^2}\right)\cos^4\alpha_1 + \left(\frac{1-k_2}{k_2^2}\right)\cos^4\alpha_2\\ &\quad - \lambda\left(\frac{d}{t}\right)\left(\frac{1-q_s\dfrac{d}{t}}{2-q_s}\right)\sin^2\alpha_2\cos^2\alpha_2\end{aligned}\right\} \quad (61l)$$

Also:

$$\frac{\cos^2\alpha_1\, f_s}{n\, f_{c1}\, k_1} = \left(\frac{1-k_1}{k_1^2}\right)\cos^4\alpha_1 + \left(\frac{1-k_2}{k_2^2}\right)\sin^4\alpha_1 - \lambda\left(\frac{d}{t}\right)\left(\frac{1-q_s\dfrac{d}{t}}{2-q_s}\right)\sin^2\alpha_1\cos^2\alpha_1 \quad (61m)$$

$$\left.\begin{aligned} j_2 &= \left(\frac{M_2}{M_1}\right)\left(\frac{j_1}{\tan^2\alpha}\right)\\ j_s &= \left(\frac{M_s}{M_1}\right)\left(\frac{d}{t}\right)\left(\frac{j_1}{\tan\alpha}\right)\end{aligned}\right\} \quad (61n)$$

$$\left.\begin{aligned} k_2 &= 3\,(1-j_2)\\ q_s &= \frac{3+6\,j_s-6\dfrac{d}{t}}{2+3\,j_s-3\dfrac{d}{t}}\end{aligned}\right\} \quad (61p)$$

If the design planes are principal planes:

$$\left.\begin{aligned} M_s &= 0\\ j_s &= 0\\ q_s &= \frac{6\dfrac{d}{t}-3}{3\dfrac{d}{t}-2}\end{aligned}\right\} \quad (61q)$$

(*B*).—*Tension Over Part of Section on One Design Plane, Compression over Entire Section on the Other Design Plane.*—In Fig. 21(*a*) is shown a section of a reinforced concrete beam on which a moment, M_2, is acting. The usual stress-strain diagram for this case is shown in Fig. 21(*b*). The stress in

the concrete acting with the stress in the steel produces a couple which balances the applied couple, M_2; but if there should be an initial tension in the steel, this stress (which is eccentric to the section), produces a moment which partly counteracts the applied couple, and at the same time produces a direct compressive stress on the section. Under certain conditions it may produce compression over the entire section as shown in Fig. 21(c). If the moment of the initial stress about the center of the section should happen to equal the applied moment, then the compression is uniformly distributed over the section, as shown in Fig. 21(d). If the initial stress moment is larger than the applied couple, the stress distribution may be as shown in Fig. 21(e).

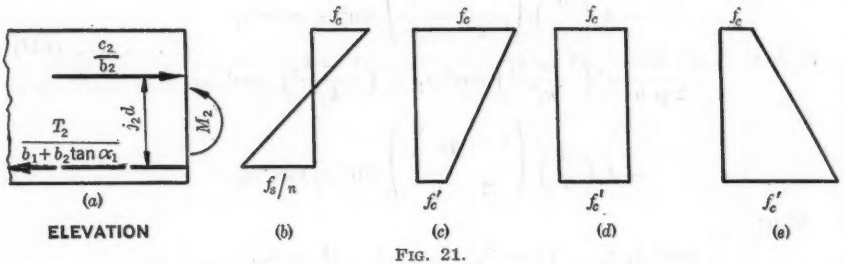


FIG. 21.

Since (as shown in Case I(A)), the stress in the steel is produced by moments on two planes, it may so happen that the moment of steel stress component on a plane may be greater than the applied couple and cause a kind of reversal of stress. The result will be similar to the case of initial stress just discussed. Case I(A) will fail when either value of k is greater than unity. For this contingency, Case I(B), or one of the following cases, must be used.

Referring to Fig. 19, the state of stress is assumed to be such that the stress-strain distribution on Plane $B\,C$ is as shown in Fig. 22(b), and that on Plane $C\,D$, as shown in Fig. 22(d).

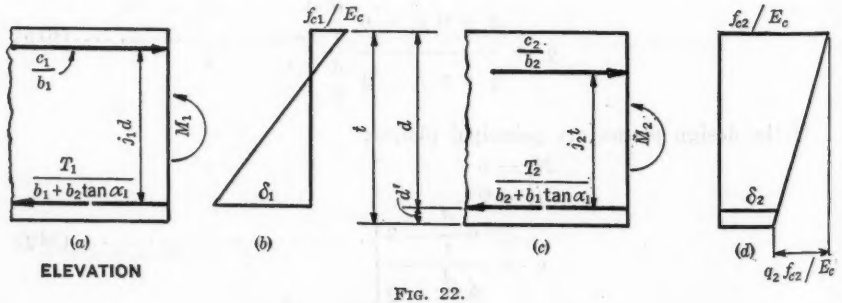


FIG. 22.

From Fig. 19 and Fig. 22:

$$\left.\begin{aligned} T &= f_s A_s (b_1 \cos \alpha_1 + b_2 \sin \alpha_1) \\ T_1 &= T \cos \alpha_1 \\ T_2 &= T \sin \alpha_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (62a)$$

$$\left.\begin{aligned} C_1 &= \frac{1}{2} f_{c1}\, k_1\, b_1\, d \\ C_2 &= \frac{1}{2} f_{c2}\, (2 - q_2)\, b_2\, t \end{aligned}\right\} \qquad (62b)$$

$$\left.\begin{aligned} P_1 &= \frac{1}{2} h_1\, (2 - q_s)\, b_1\, t \\ P_2 &= \frac{1}{2} h_2\, (2 - q_s)\, b_2\, t \end{aligned}\right\} \qquad (62c)$$

On Plane $B\,C\,F$:

$$\left.\begin{aligned} \frac{C_1}{b_1} &= \frac{T_1}{b_1 + b_2 \tan \alpha_1} \\ \frac{P_1}{b_1} &= \frac{T_2}{b_1 + b_2 \tan \alpha_1} \end{aligned}\right\} \qquad (62d)$$

On Plane $D\,C\,E$:

$$\left.\begin{aligned} \frac{C_2}{b_2} &= \frac{T_2}{b_1 \cot \alpha_1 + b_2} \\ \frac{P_2}{b_2} &= -\frac{T_1}{b_1 \cot \alpha_1 + b_2} \end{aligned}\right\} \qquad (62e)$$

$$\left.\begin{aligned} f_s\, p \cos^2 \alpha_1 &= \frac{1}{2} f_{c1}\, k_1 \\ f_s\, p \sin^2 \alpha_1 &= \frac{1}{2} f_{c2}\, (2 - q_2)\, \frac{t}{d} = f_s\, p \cos^2 \alpha_2 \\ f_s\, p \sin \alpha_1 \cos \alpha_1 &= \frac{1}{2} h_1\, (2 - q_s)\, \frac{t}{d} = \frac{1}{2} h_2\, (2 - q_s)\, \frac{t}{d} \\ f_s\, p \sin \alpha_2 \cos \alpha_2 &= \frac{1}{2} h_1\, (2 - q_s)\, \frac{t}{d} = \frac{1}{2} h_2\, (2 - q_s)\, \frac{t}{d} \end{aligned}\right\} \qquad (62f)$$

$$\left.\begin{aligned} M_1 &= \frac{1}{2} f_{c1}\, k_1\, j_1\, d^2 \\ M_2 &= \frac{1}{2} f_{c2}\, (2 - q_2)\, j_2\, t^2 \\ M_s &= \frac{1}{2} h\, (2 - q_s)\, j_s\, t^2 \end{aligned}\right\} \qquad (62g)$$

$$\left.\begin{aligned} M_1 &= f_s\, p\, j_1\, d^2 \cos^2 \alpha_1 \\ M_2 &= f_s\, p\, j_2\, d\, t \sin^2 \alpha_1 = f_s\, p\, j_2\, d\, t \cos^2 \alpha_2 \\ M_s &= f_s\, p\, j_s\, d\, t \sin \alpha_1 \cos \alpha_1 = f_s\, p\, j_s\, d\, t \sin \alpha_2 \cos \alpha_2 \end{aligned}\right\} \qquad (62h)$$

$$\left.\begin{aligned} j_1 &= 1 - \frac{k_1}{3} \\ j_2 &= \frac{d}{t} - \frac{3 - 2\, q_2}{6 - 3\, q_2} \\ j_s &= \frac{d}{t} - \frac{3 - 2\, q_s}{6 - 3\, q_s} \end{aligned}\right\} \qquad (62i)$$

From Figs. 20 and 22:

$$\left.\begin{aligned}\delta_1 &= \frac{f_{c1}}{E_c}\left(\frac{1-k_1}{k_1}\right)\\ \delta_2 &= -\frac{f_{c2}}{E_c}\left(1-q_2\frac{d}{t}\right)\\ \sigma &= -\frac{\lambda h}{E_c}\left(1-q_s\frac{d}{t}\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(62j)$$

Substituting Equations (62*j*) in Equation (60):

$$\left.\begin{aligned}\frac{f_s}{n} &= f_{c1}\left(\frac{1-k_1}{k_1}\right)\cos^2\alpha_1 - f_{c2}\left(1-q_2\frac{d}{t}\right)\sin^2\alpha_1 - \lambda h_1\left(1-q_s\frac{d}{t}\right)\sin\alpha_1\cos\alpha_1\\ \frac{f_s}{n} &= f_{c1}\left(\frac{1-k_1}{k_1}\right)\sin^2\alpha_2 - f_{c2}\left(1-q_2\frac{d}{t}\right)\cos^2\alpha_2 - \lambda h_2\left(1-q_s\frac{d}{t}\right)\sin\alpha_2\cos\alpha_2\end{aligned}\right\}(62k)$$

Eliminating f_s, f_{c1}, and f_{c2} between Equations (62*f*) and (62*k*):

$$\left.\begin{aligned}\frac{1}{2\,p\,n} &= \left(\frac{1-k_1}{{k_1}^2}\right)\cos^4\alpha_1 - \left(\frac{d}{t}\right)\left(\frac{1-q_2\frac{d}{t}}{2-q_2}\right)\sin^4\alpha_1\\ &\quad - \lambda\frac{d}{t}\left(\frac{1-q_2\frac{d}{t}}{2-q_s}\right)\sin^2\alpha_1\cos^2\alpha_1\\ \frac{1}{2\,p\,n} &= \left(\frac{1-k_1}{{k_1}^2}\right)\sin^4\alpha_2 - \left(\frac{d}{t}\right)\left(\frac{1-q_2\frac{d}{t}}{2-q_2}\right)\cos^4\alpha_2\\ &\quad - \lambda\left(\frac{d}{t}\right)\left(\frac{1-q_s\frac{d}{t}}{2-q_s}\right)\sin^2\alpha_2\cos^2\alpha_2\end{aligned}\right\}\quad\ldots\ldots(62l)$$

Also:

$$\frac{\cos^2\alpha_1\, f_s}{n\, f_{c1}\, k_1} = \left(\frac{1-k_1}{{k_1}^2}\right)\cos^4\alpha_1 - \frac{d}{t}\left(\frac{1-q_2\frac{d}{t}}{2-q_2}\right)\sin^4\alpha_1 - \lambda\left(\frac{d}{t}\right)\left(\frac{1-q_s\frac{d}{t}}{2-q_s}\right)\sin^2\alpha_1\cos^2\alpha_1\quad\ldots\ldots\ldots\ldots(62m)$$

$$\left.\begin{aligned}j_2 &= \left(\frac{d}{t}\right)\left(\frac{M_2}{M_1}\right)\left(\frac{j_1}{\tan^2\alpha}\right)\\ j_s &= \left(\frac{d}{t}\right)\frac{M_s}{M_1}\frac{j_1}{\tan\alpha_1}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(62n)$$

$$\left.\begin{aligned}q_2 &= \frac{3+6\,j_2-6\frac{d}{t}}{2+3\,j_2-3\frac{d}{t}}\\ q_s &= \frac{3+6\,j_s-6\frac{d}{t}}{2+3\,j_s-3\frac{d}{t}}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(62p)$$

If the design planes are principal planes:

$$\left.\begin{aligned} M_s &= 0 \\ j_s &= 0 \\ q_s &= \frac{6\,\frac{d}{t} - 3}{3\,\frac{d}{t} - 2} \end{aligned}\right\} \qquad (62q)$$

(*C*).—*Tension Over Part of Section on Both Design Planes But on Opposite Sides of Section.*—This case may arise in two ways: (1) recalling the discussion of Case I(*B*), if the initial tension moment is greater than the applied moment and produces a reversal of stress such that the resultant falls outside the middle third of the section; and (2) if the moments on both design planes are of opposite signs. If tension is allowed in the concrete, Case I (*B*) governs, q_2 will be greater than unity, and f_{c2} will be negative. If no tension is allowed in the concrete, or if it exceeds the allowable value, then Case I (*C*) must be used.

In Case I (*C*), the resultant of the compressive stresses falls outside the middle third and nominal steel should be used in the tension area. If f_{c2} is too high, Case 2 (*A*), or one of the following cases, will apply.

Fig. 23 shows the stress deformation diagram for the design plane on which there is tension over the extrados part of the section. The other diagrams are the same as in Figs. 21 and 22 and are not shown.

It will be noticed that Equations (62) are derived in the same way as Equations (61) and that the only differences between the corresponding equations are in the values of k, q, j, and in the expressions involving these symbols. The same thing is true for Equations (63) and the detailed derivation is not given:

$$\left.\begin{aligned} f_s\,p\cos^2\alpha_1 &= \frac{1}{2}\,f_{c1}\,k_1 \\ f_s\,p\sin^2\alpha_1 &= \frac{1}{2}\,f_{c2}\,k_2 = f_s\,p\cos^2\alpha_2 \\ f_s\,p\sin\alpha_1\cos\alpha_1 &= \frac{1}{2}\,h_1\,(2-q_s)\,\frac{t}{d} = \frac{1}{2}\,h_2\,(2-q_s)\,\frac{t}{d} \\ f_s\,p\sin\alpha_2\cos\alpha_2 &= +\frac{1}{2}\,h_1\,(2-q_s)\,\frac{t}{d} = \frac{1}{2}\,h_2\,(2-q_s)\,\frac{t}{d} \end{aligned}\right\} \qquad (63f)$$

$$\left.\begin{aligned} M_1 &= \frac{1}{2}\,f_{c1}\,k_1\,j_1\,d^2 \\ M_2 &= \frac{1}{2}\,f_{c2}\,k_2\,j_2\,d^2 \\ M_s &= \frac{1}{2}\,h_1\,(2-q_s)\,j_s\,t^2 = \frac{1}{2}\,h_2\,(2-q_s)\,j_s\,t^2 \end{aligned}\right\} \qquad (63g)$$

$$\left.\begin{aligned} M_1 &= f_s\,p\,j_1\,d^2\cos^2\alpha_1 \\ M_2 &= f_s\,p\,j_2\,d^2\sin^2\alpha_1 = f_s\,p\,j_2\,d^2\cos^2\alpha_2 \\ M_s &= f_s\,p\,j_s\,d\,t\sin\alpha_1\cos\alpha_1 = f_s\,p\,j_s\,d\,t\sin\alpha_2\cos\alpha_2 \end{aligned}\right\} \qquad (63h)$$

$$\left.\begin{aligned} j_1 &= 1 - \frac{k_1}{3} \\ j_2 &= \frac{k_2}{3} - \frac{d'}{d} \\ j_s &= \frac{d}{t} - \frac{3 - 2\,q_s}{6 - 3\,q_s} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63i)$$

From Fig. 20:

$$\left.\begin{aligned} \delta_1 &= \frac{f_{c1}}{E_c}\left(\frac{1 - k_1}{k_1}\right) \\ \delta_2 &= \frac{f_{c2}}{E_c}\left(\frac{k_2 - \dfrac{d'}{d}}{k_2}\right) \\ \sigma &= -\frac{\lambda\, h}{E_c}\left(1 - q_s \frac{d}{t}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63j)$$

$$\left.\begin{aligned} \frac{f_s}{n} &= f_{c1}\left(\frac{1 - k_1}{k_1}\right)\cos^2\alpha_1 - f_{c2}\left(\frac{k_2 - \dfrac{d'}{d}}{k_2}\right)\sin^2\alpha_1 \\ &\quad - \lambda\, h_1\left(1 - q_s\frac{d}{t}\right)\sin\alpha_1\cos\alpha_1 \\ \frac{f_s}{n} &= f_{c1}\left(\frac{1 - k_1}{k_1}\right)\sin^2\alpha_2 - f_{c2}\left(\frac{k_2 - \dfrac{d'}{d}}{k_2}\right)\cos^2\alpha_2 \\ &\quad - \lambda\, h_2\left(1 - q_s\frac{d}{t}\right)\sin\alpha_2\cos\alpha_2 \end{aligned}\right\} \quad \ldots\ldots\ldots (63k)$$

$$\left.\begin{aligned} \frac{1}{2\,p\,n} &= \left(\frac{1 - k_1}{k_1^2}\right)\cos^4\alpha_1 - \left(\frac{k_2 - \dfrac{d'}{d}}{k_2^2}\right)\sin^4\alpha_1 \\ &\quad - \lambda\left(\frac{d}{t}\right)\left(\frac{1 - q\,\dfrac{d}{t}}{2 - q_s}\right)\sin^2\alpha_1\cos^2\alpha_1 \\ \frac{1}{2\,p\,n} &= \left(\frac{1 - k_1}{k_1^2}\right)\sin^4\alpha_2 - \left(\frac{k_2 - \dfrac{d'}{d}}{k_2^2}\right)\cos^4\alpha_2 \\ &\quad - \lambda\left(\frac{d}{t}\right)\left(\frac{1 - q_s\dfrac{d}{t}}{2 - q_s}\right)\sin^2\alpha_2\cos^2\alpha_2 \end{aligned}\right\} \quad \ldots\ldots\ldots (63l)$$

Also,

$$\left.\begin{aligned} \frac{\cos^2\alpha_1 f_s}{n\, f_{c1}\, k_1} &= \left(\frac{1 - k_1}{k_1^2}\right)\cos^4\alpha_1 - \left(\frac{k_2 - \dfrac{d'}{d}}{k_2^2}\right)\sin^4\alpha_1 \\ &\quad - \lambda\left(\frac{d}{t}\right)\left(\frac{1 - q_s\dfrac{d}{t}}{2 - q_s}\right)\sin^2\alpha_1\cos^2\alpha_1 \end{aligned}\right\} \quad \ldots\ldots\ldots (63m)$$

$$\left.\begin{aligned} j_2 &= \left(\frac{M_2}{M_1}\right)\frac{j_1}{\tan^2\alpha_1} \\ j_s &= \left(\frac{d}{t}\right)\left(\frac{M_s}{M_1}\right)\frac{j_1}{\tan\alpha_1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63n)$$

$$\left.\begin{aligned} k_2 &= 3\left(j_2 + \frac{d'}{d}\right) \\ q_s &= \frac{3 + 6\,j_s - 6\,\frac{d}{t}}{2 + 3\,j_s - 3\,\frac{d}{t}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63p)$$

If the design planes are principal planes:

$$\left.\begin{aligned} M_s &= 0 \\ j_s &= 0 \\ q_s &= \frac{6\,\frac{d}{t} - 3}{3\,\frac{d}{t} - 2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63q)$$

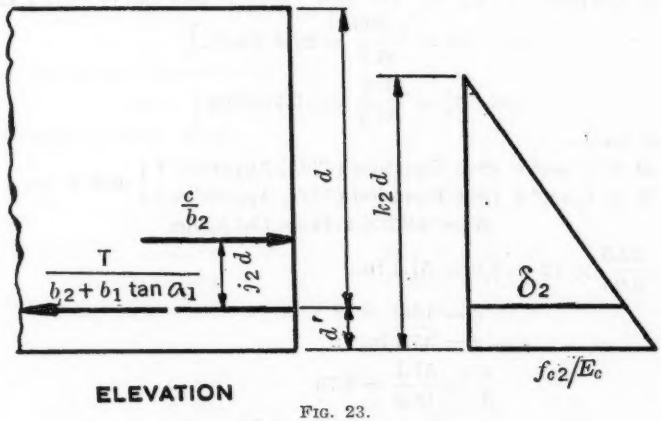


FIG. 23.

SECTION 5.—FINAL DESIGN FORMULAS

Section 4 shows the method of deriving the design formulas. In general, the stress in a strand of steel at an angle to a plane is produced by the forces acting on the plane in question and also by the forces acting on the plane at right angles to it. Furthermore, even the concrete stresses on a plane are partly dependent on the forces acting on the plane at right angles to it, and which are transmitted to it by the steel. Since two planes at right angles to each other must be analyzed together, many combinations of stress conditions on the two design planes are possible. If there are n stress conditions possible on a plane, there must be n^2 possible combinations when two planes are treated together. Hence, there would be a large number of cases to be analyzed if the design formulas were written according to the method used in the Section 4.

It is apparent from a study of Section 4, that the two design planes can be treated independently, except for the equations, (*k*), (*l*), and (*m*). There is, however, a certain symmetry about all the equations which makes it possible to treat each design plane independently and thus greatly reduce the number of cases to be analyzed. To analyze the stresses for any given set of conditions, the proper set of formulas for one design plane is combined with the proper set for the other plane. For brevity these formulas are tabulated and placed in Appendix I under the caption, "Equation (64)," in order to help the designer in referring back to the text when necessary.

Example 2.—Proportioning of Sections

Part I concluded with a numerical example which was carried up to the point at which the maximum moments, thrusts, and shears were obtained. Numerical Example 1 will now be resumed and several sections designed, thus illustrating the application of the design formulas which have been developed in Part II.

Design of Steel at Crown ($x = 0$)

$$\left.\begin{matrix} M_x = 741 \text{ kip-ft.} \\ T_x = 452 \text{ kips} \end{matrix}\right\} \quad \left.\begin{matrix} M_z = 1\,509 \text{ kip-ft.} \\ T_z = 418 \text{ kips} \end{matrix}\right\} \quad \left.\begin{matrix} t = 21 \text{ in.} \\ d = 18.5 \text{ in.} \end{matrix}\right\} \quad \left.\begin{matrix} \varepsilon = 1.192 \\ \varepsilon^2 = 1.420 \end{matrix}\right\}$$

$$\left.\begin{aligned} m &= \frac{1\,509}{66.9} = 22.5 \text{ kip-ft.} \\ t_n &= \frac{418}{66.9} = 6.25 \text{ kips} \end{aligned}\right\}$$

Fascia Steel.—

$$\left.\begin{matrix} M = m \sec^2 \theta' \text{ (See Equation (53}d\text{), Appendix I)} \\ N = t_n \sec^2 \theta' \text{ (See Equation (53}f\text{), Appendix I)} \end{matrix}\right\} \text{ and } \theta' = \theta$$

$$N = 6.25 \times 2.42 = 15.13 \text{ kips}$$

$$e = \frac{22.5}{6.25} \times 12 + 8.0 = 51.1 \text{ in.}$$

$$r = 16.0$$

$$e - r = 35.1 \text{ in.}$$

$$\frac{e}{d} = \frac{51.1}{18.5} = 2.76$$

$$K = \frac{N e}{b d^2} = \frac{15.13 \times 2.76}{12 \times 18.5} = 1.88$$

K (for tension steel only) $= 138$ (for $f_c = 800, f_s = 18\,000, n = 15$)

$$N' = \left(\frac{188 - 138}{188}\right) \times 15.13 = 4.03 \text{ kips}$$

$$f'_s = \frac{k d - d'}{k d} \times n f_c = \left(\frac{0.400 \times 18.5 - 2.5}{0.400 \times 18.5}\right) \times 12\,000 = 7.95 \text{ kips}$$

$$A'_s = \frac{N' e}{f'_s r} = \frac{4.03 \times 51.1}{7.95 \times 16.0} = 1.62 \text{ sq. in.}$$

$$\left.\begin{aligned} A_{s1} &= \frac{N'(e - r)}{f_s r} = \frac{4.03 \times 35.1}{18.0 \times 16.0} = 0.49 \\ A_{s2} &= 0.0059 \times 12 \times 18.5 = 1.31^7 \end{aligned}\right\} 1.80 \text{ sq. in.}$$

[7] Hool and Johnson, p. 405.

The fascia section extends for a distance (parallel to the abutments) of about one and one-half times the depth of the section; and where cross-steel is required, it should extend for at least a bond length of the cross-steel.

Center Steel (*Parallel and Perpendicular to Spandrel Walls*).—

$$\frac{m_v}{b} = \frac{74.1}{66.9} = 11.1 \text{ kips}$$

$$\varepsilon' m = 22.5 \times 1.192 = 26.8$$

$$m_s = \frac{1}{2} \times 37.9 = 19.0 \text{ kip-ft. (See Equation (44) Appendix I)}$$

$$m' = \frac{\varepsilon' m_v}{b} = 1.192 \times 11.1 = 13.2 \text{ kip-ft.}$$

$$\frac{t_z}{b} = \frac{452}{66.9} = 6.75 \text{ kips}$$

$$\varepsilon' t_n = 1.192 \times 6.25 = 7.45 \text{ kips}$$
$$-0.70 \text{ kips}$$
$$-0.35 \text{ kips}$$
$$-1.05 \text{ kips}$$
$$\varepsilon' t_n = 7.45 \text{ kips}$$
$$t_s = 6.40 \text{ kips}$$
(See Equation (35), Appendix I)

$m = 22.5$ kip-ft. $t_n = 6.25$ kips
$m' = 13.2$ kips $t_n = 7.63$ kips
$m_s = 19.0$ $t_s = 6.40$ kips

$$t'_n = \varepsilon' t_s = 1.192 \times 6.40 = 7.63 \text{ kips}$$

(*a*) Minimum Cross Steel.—

$$\psi = 180 - \theta'$$

$$\sin \theta' = 0.766 \quad \sin^2 \theta' = 0.587$$
$$\cos \theta' = 0.643 \quad \cos^2 \theta' = 0.413$$

$m \sin^2 \theta' =$	$22.5 \times 0.587 =$	13.2
$m \cos^2 \theta' =$	$13.2 \times 0.413 =$	5.3
$-2 m_s \sin \theta' \cos \theta' =$	$37.9 \times 0.766 \times 0.643 =$	-18.6
	$M =$	0

$t_n \sin^2 \theta' =$	$6.25 \times 0.587 =$	3.7
$t'_n \cos^2 \theta' =$	$7.63 \times 0.413 =$	3.2
$-2 t_s \sin \theta \cos \theta =$	$2 \times 6.40 \times 0.492 =$	-6.4
	$N =$	0.5 kip (negligible)

The foregoing computations show that only nominal steel is required.

(*b*) Minimum Longitudinal Steel.—

$m \cos^2 \theta' = 22.5 \times 0.413 =$	9.2	$t_n \cos^2 \theta' = 6.25 \times 0.413 =$	2.6
$m' \sin^2 \theta' = 13.2 \times 0.587 =$	7.8	$t'_n \sin^2 \theta' = 7.63 \times 0.587 =$	4.5
$2 m_s \sin \theta \cos \theta$	18.6	$2 t_s \sin \theta' \cos \theta' =$	6.4
	35.6 kip-ft.		13.5 kips

$$e = \frac{35.6}{13.5} \times 12 + 8.0 = 39.7$$

$$\frac{e}{d} = \frac{39.7}{18.5} = 2.14$$

$$K = \frac{13.5 \times 2.14}{12 \times 18.5} = 130$$

$$f_c = 770 \text{ lb.}$$
$$f_s = 18\,000$$
$$p = 0.0050$$

$$A_s = 12 \times 18.5 \times 0.0050 = 1.11 \text{ sq. in.}$$

(*c*) Principal Planes.—

$$\tan 2\psi = \frac{2\,m_s}{m' - m} = -\frac{37.9}{9.2}$$

$$\tan (2\psi - 90) = \frac{9.2}{37.9} = 0.243$$

$2\psi_1 - 90 = 13° 40'$; $\psi_1 = 51° 50'$; and $\alpha_1 = \psi_1 + \theta' - 90 = 11° 50'$

$$\sin \psi_1 = 0.786 \quad \sin^2 \psi_1 = 0.618$$
$$\cos \psi_1 = 0.618 \quad \cos^2 \psi_1 = 0.382$$
$$\sin \psi_1 \cos \psi_1 = 0.485$$

$$m \sin^2 \psi_1 = 22.5 \times 0.618 = 13.90 \text{ kip-ft.}$$
$$m' \cos^2 \psi_1 = 13.2 \times 0.382 = 5.05$$
$$2\,m_s \sin \psi_1 \cos \psi_1 = 37.9 \times 0.485 = 18.40$$
$$M_1 = 37.35 \text{ kip-ft.}$$

$$t_n \cos^2 \psi_1 = 6.25 \times 0.618 = 3.86 \text{ kips}$$
$$t'_n \sin^2 \psi_1 = 7.63 \times 0.382 = 2.91$$
$$2\,t_s \sin \psi_1 \cos \psi_1 = 12.8 \times 0.485 = 6.21$$
$$N_1 = 12.98 \text{ kips}$$

$$m \sin^2 \psi_2 = 22.5 \times 0.382 = 8.60 \text{ kip-ft.}$$
$$m' \cos^2 \psi_2 = 13.2 \times 0.618 = 8.15$$
$$2\,m_s \sin \psi_2 \cos \psi_2 = -18.40$$
$$M_2 = -1.65 \text{ kip-ft.}$$

$$t_n \cos^2 \psi_2 = 6.25 \times 0.382 = 2.39 \text{ kips}$$
$$t'_n \sin^2 \psi_2 = 7.63 \times 0.618 = 4.71$$
$$2\,t_s \sin \psi_2 \cos \psi_2 = -6.21$$
$$N_2 = 0.89 \text{ kip}$$

The foregoing computations show that M_2 and N_2 are negligible.

Normal Steel Plane 1.

$$e = \frac{37.4}{13.0} \times 12 + 8.0 = 42.2 \text{ in.}$$

$$\frac{e}{d} = \frac{42.2}{18.5} = 2.28$$

$$K = \frac{13.0 \times 2.28}{12 \times 18.5} = 1.34$$

$F_{c1} = 770$ lb.
$F_s = 19\,200$ lb. (trial value)
$P = 0.00493$

Normal Steel Plane 2.
(None required)

For Plane 1, use Case IV (A) (Equations (64)) and for Plane 2, use Case I (B):

$$\sin \alpha_1 = 0.205 \quad \sin^2 \alpha_1 = 0.042$$
$$\cos \alpha_1 = 0.979 \quad \cos^2 \alpha_1 = 0.958$$
$$\sin \alpha_1 \cos \alpha_1 = 0.200$$

$$f_s\, p \cos^2 \alpha_1 = \frac{1}{2} f_{c1}\, k_1 \frac{-N_1}{b\, d} = F_s\, P = 19\,200 \times 0.00493 = 94.5$$

$$f_s\, p = \frac{94.5}{0.958} = 98.7$$

On Plane 2:

$$M_2 = 0, \therefore j_2 = 0, k_2 = 3\left(j_2 + \frac{d'}{d}\right) = \frac{3 \times 2.50}{18.5} = 0.405$$

$$a_2 = -\left(\frac{k - \frac{d'}{d}}{k}\right) = -\frac{2}{3}$$

$$f_s\, p \cos^2 \alpha_1 = \frac{1}{2} f_{c2}\, k_2$$

$$f_{c2} = \frac{2 \times 98.7 \times 0.42}{0.405} = 20.4 \text{ lb. per sq. in.}$$

On Plane 1:

$$\frac{d}{t} = \frac{18.5}{21.0} = 0.882$$

$$q_s = \frac{3 - \frac{6\, d}{t}}{2 - \frac{3\, d}{t}} = 3.54$$

$$a_s = -\left(1 - q_s \frac{d}{t}\right) = -(1 - 0.882 \times 3.54) = 2.12$$

$$h_1 = \frac{2\, f_s\, p\, b\, d \sin \alpha_1 \cos \alpha_1 + S}{b\, t\, (2 - q_s)};\ S = 0 \text{ for principal planes}$$

$$h_1 = 2 \frac{d}{t} \times \frac{f_s\, p \sin \alpha_1 \cos \alpha_1}{(2 - q_s)} = \frac{2 \times 0.882 \times 98.7 \times 0.200}{-1.58} = -22 \text{ lb.}$$

$$\frac{f_s}{n} = \Sigma\, f_c\, a \cos^2 \alpha + \lambda\, h\, a_s \sin \alpha \cos \alpha$$

$$n\, f_c \left(\frac{1 - k_1}{k_1}\right) = F_s$$

$$f_s = 19\,200 \times 0.958 + 15 \times 20.4 \left(-\frac{2}{3}\right) 0.042$$
$$+ 37.5 \times (-22.0)(+2.12) \times 0.200$$
$$f_s = 18\,400 + (-8) + (-350) = 18\,050 \text{ lb.}$$

$$p = \frac{f_s\, p}{f_s} = \frac{98.7}{18\,000} = 0.00543$$
$$A_s = 0.00543 \times 222 = 1.20 \text{ lb. per sq. in.}$$

Maximum Horizontal Shear.—

$$\tan 2\psi = \frac{m - m'}{2\, m_s} = \frac{9.2}{37.9} = 0.243$$

$$\psi = 6^\circ\ 50' \text{ and } \alpha_1 = (6^\circ\ 50') + 50^\circ\ 90^\circ = -(33^\circ\ 10')$$

$$\left.\begin{matrix}\sin\psi_1 = 0.119 \\ \cos\psi_1 = 0.993\end{matrix}\right\} \quad \left.\begin{matrix}\sin^2\psi_1 = 0.014 \\ \cos^2\psi_1 = 0.985\end{matrix}\right\} \quad \left.\begin{matrix}\sin\alpha_1 = -0.547 \\ \cos\alpha_1 = 0.837\end{matrix}\right\}$$

$$\sin^2\psi_1 - \cos^2\psi_1 = -0.971 \qquad \sin\alpha_1 \cos\alpha_1 = -0.458$$

$$\sin\psi_1 \cos\psi_1 = 0.118$$

$$(m' - m)\sin\psi_1 \cos\psi_1 = (-9.2)\,(0.118) = -1.08$$

$$m_s\,(\sin^2\psi_1 - \cos^2\psi_1) = 19.0 \times (-0.971) = -18.48$$

$$M_s = -19.56 \text{ kip-ft.}$$

$$(t'_n - t_n)\sin\psi_1 \cos\psi_1 = (-1.38)\,(0.118) = -0.16$$

$$t_s\,(\sin^2\psi_1 - \cos^2\psi_1) = 6.40 \times (-0.971) = -6.22$$

$$-6.38 \text{ kips}$$

$$f_s\, p \sin\alpha_1 \cos\alpha_1 = \frac{1}{2}\, h\,(2 - q_s)\,\frac{t}{d} - \frac{s}{b\,d}$$

$$f_s\, p \sin\alpha_1 \cos\alpha_1 = 98.7 \times (-0.458) = -45.2$$

$$\frac{S}{b\,d} = \frac{-6380}{12 \times 18.5} = \frac{-29.7}{-74.9}$$

$$\frac{1}{2}\, h\,(2 - q_s) = \frac{d}{t} \times (-74.9) = -66.0$$

$$e_s\,\frac{-19.56}{-6.38} \times 12 + 8.0 = 44.9 \text{ in.}$$

$$K_s = \frac{S\, e_s}{b\, t^2} = \frac{(-6.38) \times 44.9}{12 \times 21 \times 21} = -54.2$$

$$\left.\begin{matrix}\dfrac{K_s}{j_s} = \dfrac{1}{2}\, h\,(2 - q_s) = -\dfrac{54.2}{j_s} \\ \dfrac{1}{2}\, h\,(2 - q_s) = -66.0\end{matrix}\right\} \quad \begin{matrix}\dfrac{54.2}{j_s} = 66.0 \\ j_s = 0.820\end{matrix}$$

$$q_s = \frac{3 + 6\, j_s - 6\,\dfrac{d}{t}}{2 + 3\, j_s - 3\,\dfrac{d}{t}} = \frac{3 + 6\,(0.820 - 0.882)}{2 + 3\,(0.820 - 0.882)} = 1.438$$

$$h = \frac{2 \times (-66.0)}{2 - q_s} = \frac{-132.0}{0.562} = -235 \text{ lb. ((O.K.), 240 lb. allowed)}$$

$$h' = (1 - q_s)\, h = (1 - 1.438)\,(-235) = 103 \text{ lb. (O.K.)}$$

Maximum Radial Shear.—

$$V = \frac{t_u}{b} - \frac{8}{5}\,\frac{m_v - \varepsilon' m_z}{b\, t}$$

$$V = \frac{38}{66.9} - \frac{8}{5}\,\frac{(-9.2)}{21 \times 66.9} = 0.58 \text{ kip.}$$

$$V_{\max.} = 0.580 \times 2.42 = 1.4 \text{ kips}\,;\ v = \frac{1400}{0.8 \times 12 \times 18.5} = 7.9 \text{ lb. (O.K.)}$$

Design of Steel at the Section, $x = 20.6$

$$t = 54 \text{ in.}$$

$$\varepsilon' = \varepsilon = 1.192 \qquad \varepsilon'^2 = 1.420$$

$$\left.\begin{matrix} m_z = -5\,930 \text{ kip-ft.} \\ t_v = \quad 803 \text{ kips} \end{matrix}\right\} \quad \left.\begin{matrix} m_v = -6\,740 \text{ kip-ft.} \\ t_z = \quad 667 \text{ kips} \end{matrix}\right\}$$

Center Stresses.—

$$\left.\begin{matrix} m = -\ 88.6 \text{ kip-ft.} \\ m' = -120.0 \quad \text{“} \\ m_s = -103.3 \quad \text{“} \end{matrix}\right\} \quad \left.\begin{matrix} t_n = 12.0 \ \text{kips} \\ t_n = \ 6.56 \ \text{“} \\ t_s = \ 5.5 \ \text{“} \end{matrix}\right\}$$

Fascia Steel.—

$$m = -88.6 \sec^2 \theta$$

$$t_n = 12.0 \sec^2 \theta$$

$$e' = \frac{-88.6}{12.0} \times 12 - 24.5 = -113.1 \text{ in.}$$

$$\frac{c'}{d} = \frac{113.1}{51.5} = -2.20$$

$$K = \frac{12.0 \times (+2.20) \times 2.42}{12 \times 51.5} = +103.5$$

$$f_c = 650$$

$$f'_s = 18\,000$$

$$p' = 0.0039$$

$$A'_s = 0.0039 \times 51.5 \times 12 = 2.41 \text{ lb. per sq. in.}$$

Center Steel.—

(*a*) Minimum Cross-Steel.—

$$\psi = 180 - \theta$$

$$m \sin^2 \theta = -88.6 \times 0.587 = -52.0$$

$$m' \cos^2 \theta = -120.0 \times 0.413 = -49.5$$

$$-2\, m_s \sin \theta \cos \theta = 206.6 \times 0.492 = \ 101.5$$

$$0$$

$$t_n \sin^2 \theta = 12.0 \times 0.587 = \ 7.05$$

$$t'_n \cos^2 \theta = 6.56 \times 0.413 = \ 2.71$$

$$-2\, t_s \sin \theta \cos \theta = 11.0 \times 0.492 = -5.41$$

$$N = 4.35 \text{ kips}$$

The foregoing computations show that no steel is required.

(*b*) Minimum Longitudinal Steel.—

$$m \cos^2 \theta = -88.6 \times 0.413 = -36.6$$

$$m' \sin^2 \theta = -120.0 \times 0.587 = -70.4$$

$$2\, m_s \sin \theta \cos \theta = -101.5$$

$$M = -208.5 \text{ kip-ft.}$$

$$t_n \sin^2 \theta = 12.0 \times 0.413 = \ 4.96$$

$$t'_n \cos^2 \theta = 6.56 \times 0.587 = \ 3.84$$

$$2\, t_s \sin \theta \cos \theta = \ 5.41$$

$$N = 14.21 \text{ kips}$$

$$f_c = 600$$
$$f'_s = 18\,000$$
$$p' = 0.43\%$$
$$A'_s = 2.65 \text{ lb. per sq. in.}$$

(*c*) Principal Planes (Case V).—

$$\tan 2\psi = \frac{2 m_s}{m' - m} = \frac{-206.6}{-31.4} = 6.57$$

$$\psi_1 = 40° 40'$$

$$M_1 = -208.9 \qquad N_1 = 14.30$$
$$M_2 = 0 \qquad N_2 = 4.24 \text{ (negligible)}$$

$$\alpha_1 = \psi_1 + \theta - 90 = 0° 40' \text{ (which is negligible)}$$

The design is for normal steel which gives the same condition as for minimum longitudinal steel.

Maximum Horizontal Shear.—

$$\tan 2\psi = \frac{m' - m_s}{2 m_s} = 0.152$$

$$\psi_1 = 4° 20' \qquad \alpha_1 = (4° 20') + 50° - 90° = -(35° 40')$$

$$\left.\begin{aligned} M_s &= 103.3 \text{ kip-ft.} \\ S &= -5.5 \text{ kips} \end{aligned}\right\} \qquad \frac{d}{t} = 0.973 \qquad \frac{d'}{t} = 0.027$$

$$e'_s = \frac{103.3}{-5.5} \times 12 - 24 = -250$$

$$f'_s p' \sin \alpha_1 \cos \alpha_1 = \frac{1}{2} h (2 - q_s) \frac{t}{d} - \frac{S}{b d}$$

$$f'_s p' \sin \alpha_1 \cos \alpha_1 = 18\,000 \times 0.43 \times (-0.473) = -36.6 \text{ lb.}$$

$$\frac{S}{b d} = -\frac{5\,500}{54.5 \times 12} = \frac{-8.4}{-45.0} \text{ lb.}$$

$$\frac{1}{2} h (2 - q_s) = -45.0 \times 0.973 = -43.8$$

$$K_s = \frac{S e'_s}{b t^2} = \frac{(-5.50)(-250)}{12 \times 54.0 \times 54.0} = +39.2$$

$$\frac{1}{2} h (2 - q_s) = -\frac{K_s}{j_s} = \frac{-39.2}{j_s} = -43.8$$

$$j'_s = \frac{-39.2}{-43.8} = +0.900$$

$$q_s = \frac{3 - 6 j_s - 6 \dfrac{d'}{t}}{2 - 3 j_s - 3 \dfrac{d'}{t}} = \frac{-2.562}{-0.781} = 3.28$$

$$h = \frac{-43.8 \times 2}{-1.28} = 68.5 \text{ lb.}$$

$$h' = (1 - q_s) h = -2.28 \times 68.5 \text{ lb.} = -156 \text{ lb. (O. K.)}$$

Section at Top of Post (Omitted from Table 8).

$m_x = -6\,365$ kip-ft. $\quad m_v = -1\,554$ kip-ft. $\quad t = 36$ in. $\quad \varepsilon' = \varepsilon \cos 90° = 0$

$t_v = 1\,181$ kips $\quad t_z = 630$ kips $\quad d = 33.5$ in. $\quad \sec^2 \theta' = 1$

$m = -95.0$ kip-ft. $\quad t_n = 17.8$ kips

$m' = 0 \quad t'_n = 0$ } at center

$m_s = -10.6$ kip-ft. $\quad t_s = 14.1$ kips

Fascia Steel.—

$M = m$, $N = t_n$; $f_c = 660$ lb., $f'_s = 18\,000$ lb., $p' = 0.0042$

$A_s = 0.0042 \times 12 \times 33.5 = 1.69$ lb. per sq. in.

Center Steel.—

(*a*) Minimum Cross Steel.—

$$\psi = 180 - \theta' = 180°$$

$M = 0$, $N = 0$ (no steel required).

(*b*) Minimum Longitudinal Steel.—

$M = m$, $N = t_n$ (same as fascia steel).

(*c*) Principal Planes.—

$$\tan 2\psi = \frac{2 m_s}{m' - m} = \frac{-21.2}{95.0} = -0.224$$

$$\tan (180 - 2\psi) = 0.224$$

$$\psi = 83° 30'$$

$M_1 = -96.7$ kip-ft. $\quad M_2 = 1.22$ kip-ft.

$N_1 = 21.0$ kips $\quad N_2 = 3.07$ kips $\quad \alpha_1 = (83° 30') - 90° = -(6° 30')$

This computation shows that the same steel as the fascia steel should be used.

Maximum Horizontal Shear.—Shear is "O. K." by inspection.

Example of Design for Steel in Two Directions

Assume: $M_1 = 18.0$ kip-ft. $\quad N_1 = 12.32$ kips $\quad \alpha_1 = 20° 25'$

$M_2 = 36.5$ kip-ft. $\quad N_2 = 2.68$ kips $\quad \alpha_2 = 110° 25'$

$t = 21.0$ in., $d = 18.5$ in.

Minimum steel..0.18% longitudinal

Minimum steel..0.63% cross

Normal Steel, Plane 1.

$K = 77$, $\frac{e}{d} = 1.38$

$F_{s1} = 19\,000$ lb. (trial value)

$F_{c1} = 575$ lb.

$P_1 = 0.18\%$

Normal Steel, Plane 2.

$K = 112$, $\frac{e}{d} = 9.28$

$F_{s2} = 16\,500$ lb. (trial value)

$F_{c2} = 710$ lb.

$P_2 = 0.70\%$

For the following computations use Equations (64), Case X (*A*).

$$\sum f_s p\, b\, d \cos^2 \alpha = \frac{1}{2} f_c\, k\, b\, d - N = F_s P\, b\, d$$

$$\left.\begin{aligned} 0.879\, f_\alpha\, p_\alpha + 0.121\, f_\beta\, p_\beta &= 19\,500 \times 0.0017 = 33.2 \\ 0.121\, f_\alpha\, p_\alpha + 0.879\, f_\beta\, p_\beta &= 16\,500 \times 0.007 = 115 \end{aligned}\right\}$$

Solving,

$$\begin{cases} f_\alpha\, p_\alpha = 20 \\ f_\beta\, p_\beta = 129 \end{cases}$$

from which, $q_s = 3.58$; $h = 40$ lb.; and $a_s = -2.16$

$$\lambda\, n\, h\, a_s \sin \alpha_1 \cos \alpha_1 = -1\,050 \text{ lb.}$$

$$f_s = \sum f_c\, a \cos^2 \alpha + \lambda\, h\, a_s \sin a_1 \cos \alpha.$$

$$f_\alpha = 19\,500 \times 0.879 + 16\,500 \times 0.121 - 1\,050 = 18\,100 \text{ lb.}$$

$$f_\beta = 19\,500 \times 0.121 + 16\,500 \times 0.879 + 1\,050 = 17\,850 \text{ lb.}$$

$$p_\alpha = \frac{20}{18\,000} = 0.11\%$$

$$p_\beta = \frac{129}{18\,000} = 0.64\%$$

Summary

In Part II, the theory for proportioning sections has been developed. Equations have been derived for (*a*) stress distribution; (*b*) transformation formulas; and (*c*) design formulas for skew reinforced concrete sections. The application of these formulas is illustrated by the preceding numerical examples. This part of the paper, together with Part I, gives the designer a complete working text for designing skew arches.

PART III.—APPROXIMATE ANALYSIS FOR CROWN STRESSES IN A SYMMETRICAL SKEW RIGID FRAME

The rigid frame is a special type of arch and is analyzed as an arch, but the shape of the frame is such that the angle, ϕ (as previously defined), is either nearly 0°, or nearly 90 degrees. From the analyses of two skew rigid frames, the writer found that a considerable variation in the value of the angle, ϕ, produced negligible changes in the crown stresses. These observations lead to the following approximations:

1.—(*a*) $\phi = 0°$ for the barrel of the frame.
(*b*) $\phi = 90°$ for the post of the frame.

2.—The effect of the curvature of the barrel of the frame can be neglected in places where other characteristics of the frame affect the magnitude of the stress.

These approximations have the effect of transforming the actual frame shown in Fig. 24 into the frame shown in Fig. 25. It should be noted that M_y is a direct result of the curvature of the neutral axis (that is, M_y would be zero if the neutral axis were truly straight), and the accurate analysis must be used to compute it. Fortunately, this is a simple matter. Furthermore, $M_y = 0$ for symmetrical loads and temperature variation. Making these approximations results in the following.

Approximate Final Equations for Symmetrical Two-Hinged Rigid Frames

Let T'_x equal the crown thrust of a right rigid frame produced by two parallel planes cutting the skew arch perpendicular to the abutments. All other symbols are as previously defined.

For Vertical Loading.—

$$\left.\begin{aligned} T_x &= T'_x \\ T_z &= \varepsilon\, T_x \\ M_x &= \frac{\varepsilon\left[\sum_B^P x\,\frac{s}{I} - x'\sum_B^P \frac{s}{I}\right]}{2\sum_B^C \frac{s}{I}} \\ M_z,\ T_y &= \text{same as for right arch} \\ M_y &= \frac{f - C\,T_y}{x},\ \text{as given in accurate theory} \end{aligned}\right\} \quad \ldots\ldots\ldots (65A)$$

For Horizontal Loading.—

$$\left.\begin{aligned} T_x &= T'_x \\ T_z &= \varepsilon\, T_x \\ M_x &= 0 \\ T_y,\ M_z &= \text{same as for right arch} \\ M_y &= \frac{f - C\,T_y}{x},\ \text{as given in accurate theory} \end{aligned}\right\} \quad \ldots\ldots\ldots (65B)$$

For Temperature Stresses.—

$$\left.\begin{aligned} T_x &= T'_x \sec^2\theta \\ T_z &= \varepsilon\left(T_x + \frac{1.34\,E\,c\,t\,L}{\sum_A^B x^2\,\frac{s}{I}}\right) \\ M_x &= 0 \\ M_y &= 0 \\ M_z &= -\,h\,T_x \\ T_y &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (65C)$$

The origin for x in these equations is at the center.

Derivation of Approximate Coefficients

(See Equation (19) and Fig. 25).

For All Loading.—

From A to B ($\phi = 90$; $u = x$):

$$\left.\begin{aligned} K_{11} &= \sum_A^B y^2\,\frac{s}{I} - 2\,h\sum_A^B y\,\frac{s}{I} + h^2\sum_A^B \frac{s}{I} + \varepsilon^2\sum_A^B x^2\,\frac{\lambda\, s}{F} \\ K_2 &= \varepsilon\sum_A^B x^2\,\frac{\lambda\, s}{F};\ K_3 = 0;\ K_5 = \sum_A^B x^2\,\frac{\lambda\, s}{F};\ K_6 = 0 \\ K_7 &= 0 \end{aligned}\right\} \quad \ldots (66A)$$

From B to C ($\phi = 0$; $u = y = 0$):

$$\left.\begin{aligned} K_{11} &= h^2 \sum_B^C \frac{s}{I} \\ K_2 &= 0;\ K_3 = 0;\ K_5 = 0;\ K_6 = 0 \\ K_7 &= \sum_B^C \frac{\lambda\, s}{F} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (66B)$$

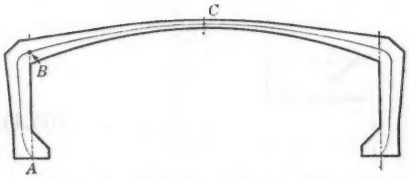


FIG. 24.—TYPICAL "RIGID FRAME."

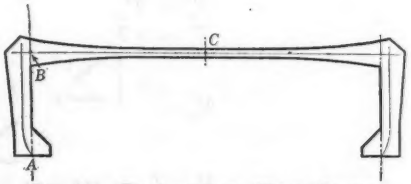


FIG. 25.—TRANSFORMED "RIGID FRAME."

For Vertical Loading.—

From A to B ($\phi = 90$; $u = x$):

$$\left.\begin{aligned} C_{11} &= \frac{1}{2}\left[(L - x')\left(-\sum_A^B y\,\frac{s}{I} + h\sum_A^B \frac{s}{I}\right) + \sum_A^B x\,y\,\frac{s}{I}\right. \\ &\qquad \left. - x'\sum_A^B y\,\frac{s}{I} - h\sum_A^B x\,\frac{s}{I} + h\,x'\sum_A^B \frac{s}{I}\right] \\ C_2 &= C_3 = C_6 = 0 \end{aligned}\right\} \ldots (67A)$$

From B to C ($\phi = 0$; $u = y = 0$):

$$\left.\begin{aligned} C_{11} &= \frac{1}{2}\left[(L - x')\left(h\sum_B^C \frac{s}{I}\right) - h\sum_B^P x\,\frac{s}{I} + h\,x'\sum_B^P \frac{s}{I}\right] \\ C_2 &= 0 \\ C_3 &= \frac{1}{2}\left[\varepsilon\sum_B^P x\,\frac{\lambda\, s}{F} - \varepsilon\, x'\sum_B^P \frac{\lambda\, s}{F}\right] \\ C_6 &= 0 \end{aligned}\right\} \ldots (67B)$$

For Horizontal Loading.—

From A to B ($\phi = 90$; $u = x$):

$$\left.\begin{aligned} C_{11} &= \frac{1}{2}\left[(h - y')\left(-\sum_A^B y\,\frac{s}{I} + h\sum_A^B \frac{s}{I}\right) + \sum_A^P y^2\,\frac{s}{I}\right. \\ &\qquad \left. - y'\sum_A^P y\,\frac{s}{I} - h\sum_A^P y\,\frac{s}{I} + h\,y'\sum_A^P \frac{s}{I}\right] \\ C_2 &= 0;\ C_3 = 0;\ C_6 = 0 \end{aligned}\right\} \ldots (68A)$$

From B to C ($\phi = 0$; $u = y = 0$):

$$\left.\begin{aligned} C_{11} &= \frac{1}{2}\left[(h - y')\left(h\sum_B^C \frac{s}{I}\right)\right] \\ C_2 &= 0;\ C_3 = 0;\ C_6 = 0 \end{aligned}\right\} \ldots\ldots\ldots (68B)$$

For Temperature Stresses.—

$$C_{11} = E\,c\,t\,L;\ C_2 = \varepsilon\,E\,c\,t\,L;\ C_3 = C_4 = C_5 = C_6 = 0 \ldots\ldots (69)$$

Approximate Coefficients

Combining the two parts of Equations (66), (67), and (68), the approximate coefficients become:

For All Loading.—

$$K_{11} = \sum_A^B y^2 \frac{s}{I} - 2h \sum_A^B y \frac{s}{I} + h^2 \sum_A^C \frac{s}{I} + \varepsilon^2 \sum_A^B x^2 \frac{s}{I}$$

$$K_2 = -\varepsilon \sum_A^B x^2 \frac{\lambda s}{F};\ K_3 = 0;\ K_5 = \sum_A^B x^2 \frac{\lambda s}{F};\ K_6 = 0;\ K_7 = \sum_B^C \frac{\lambda s}{F}$$

For Vertical Loading.—

$$C_{11} = \frac{1}{2}\left[(L - x')\left(-\sum_A^B y \frac{s}{I} + h \sum_A^C \frac{s}{I}\right) + \sum_A^B x y \frac{s}{I} - x' \sum_A^B y \frac{s}{I} - h \sum_A^P x \frac{s}{I} + h x' \sum_A^P \frac{s}{I}\right]$$

$$C_2 = 0$$

$$C_3 = \frac{1}{2}\varepsilon\left[\sum_B^P x \frac{\lambda s}{F} - x' \sum_B^P \frac{\lambda s}{F}\right]$$

For Horizontal Loading.—

$$C_{11} = \frac{1}{2}\left[(h - y')\left(-\sum_A^B y \frac{s}{I} + h \sum_A^C \frac{s}{I}\right) + \sum_A^P y^2 \frac{s}{I} - y' \sum_A^P y \frac{s}{I} - h \sum_A^P y \frac{s}{I} + h y' \sum_A^P \frac{s}{I}\right]$$

$$C_2 = 0;\ C_3 = 0$$

For Temperature.—

$$C_1 = E\, c\, t\, L$$
$$C_2 = \varepsilon\, E\, c\, t\, L$$
$$C_3 = 0$$

Derivation of Final Equations

For Vertical Loading.—Substituting the foregoing coefficients in Equation (19*b*) (Part I):

$$T_z = \frac{K_2 T_x}{K_5} = \frac{\varepsilon \sum_A^B x^2 \frac{\lambda s}{F}}{\sum_A^B x^2 \frac{\lambda s}{F}} T_x$$

therefore,

$$T_z = \varepsilon\, T_x$$

Substituting this value of T_z and the foregoing coefficients in Equations (19*a*) (Part I):

$$T_x = \frac{C_{11}}{K_{11} + \varepsilon K_2} = \frac{C_{11}}{K'_{11}}$$

in which,

$$K'_{11} = \sum_A^B y^2 \frac{s}{I} - 2h \sum_A^B y \frac{s}{I} + h^2 \sum_A^C \frac{s}{I}$$

therefore,

$$T_x = T'_x$$

Similarly, from Equations (19*c*) (Part I):

$$M_x = \frac{K_3}{K_7} = \frac{1}{2} \frac{\varepsilon \left[\sum_B^P x \frac{\lambda s}{F} - x' \sum_B^P \frac{\lambda s}{F} \right]}{\sum_B^C \frac{\lambda s}{F}}$$

but,

$$\frac{\lambda}{F} = \frac{3.58 \lambda}{b t^3} = \frac{3.58 \lambda}{12 I}$$

Substituting this value of $\frac{\lambda}{F}$ and cancelling out the constants:

$$M_x = \frac{\varepsilon \left[\sum_B^P x \frac{s}{I} - x' \sum_B^P \frac{s}{I} \right]}{2 \sum_B^C \frac{s}{I}}$$

For Horizontal Loading.—Similarly:

$$T_x = T'_x; \; T_z = \varepsilon T_x; \text{ and } M_x = 0$$

For Temperature Stresses.—Substituting the approximate coefficients in Equations (19*a*) and (19*b*):

$$\left.\begin{aligned} K_{11} T_x + K_2 T_z &= C_{11} \\ K_2 T_x + K_5 T_z &= C_2 = \varepsilon C_{11} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (70)$$

Solving:

$$T_x = \frac{C_{11} K_5 - \varepsilon C_{11} K_2}{K_{11} K_5 - K_2^2}$$

but,

$$K_5 = -\frac{K_2}{\varepsilon}$$

Therefore,

$$T_x = \frac{C_{11} (1 + \varepsilon^2)}{K_{11} + \varepsilon K_2} = \frac{C_{11}}{K'_{11}} \sec^2 \theta$$

or,

$$T_x = T'_x \sec^2 \theta$$

From the second of Equations (70):

$$T_z = \frac{\varepsilon C_{11} - K_2 T_x}{K_5} = \frac{\varepsilon E c t L}{\sum_A^B x^2 \frac{\lambda s}{F}} + \varepsilon T_x$$

but,

$$\frac{\lambda}{F} = \frac{3.58 \times 2.5}{12 I} = \frac{1}{1.34 I}$$

Therefore,

$$T_z = \varepsilon \left[T_x + \frac{1.34 E c t L}{\sum_A^B x^2 \frac{s}{I}} \right]$$

Similarly, from Equations (19*c*), $M_x = 0$.

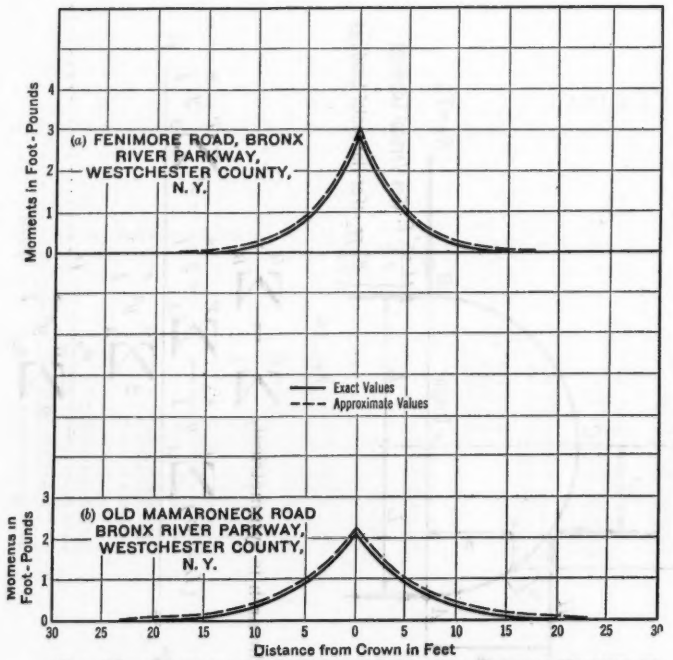

(a) FENIMORE ROAD, BRONX RIVER PARKWAY, WESTCHESTER COUNTY, N. Y.
Moments in Foot-Pounds
Exact Values
Approximate Values
(b) OLD MAMARONECK ROAD BRONX RIVER PARKWAY, WESTCHESTER COUNTY, N. Y.
Moments in Foot-Pounds
Distance from Crown in Feet

FIG. 26.—COMPARATIVE INFLUENCE LINES FOR TORSION MOMENT BY EXACT AND APPROXIMATE EQUATIONS.

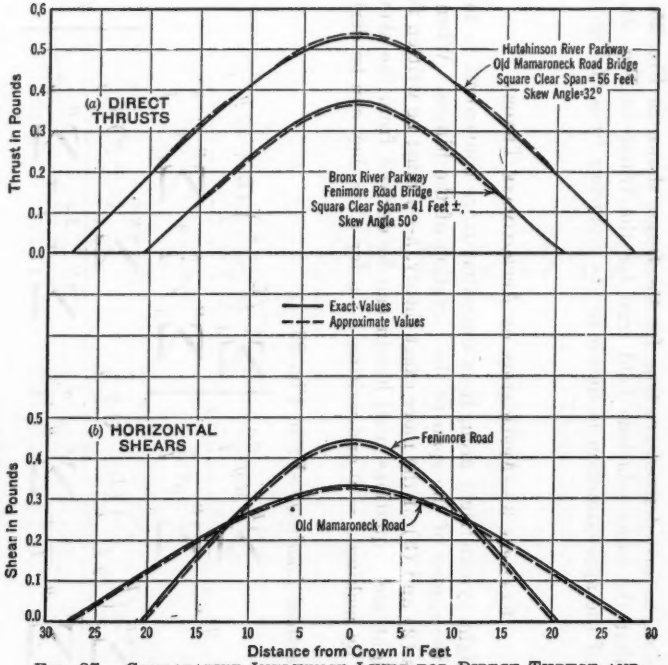


FIG. 27.—COMPARATIVE INFLUENCE LINES FOR DIRECT THRUST AND HORIZONTAL SHEAR BY EXACT AND APPROXIMATE EQUATIONS.

The relations just derived are true for the frame shown in Fig. 25. It is assumed that the same relations hold true for the frame in Fig. 24. The closeness of the approximation is shown by the influence lines of Fig. 26 and Fig. 27.

Example 3.—Application of Approximate Theory

In a right symmetrical arch, it is somewhat more convenient to take the origin at the center of the span at the elevation of the hinges. When using Equations (2) and (3) of the more exact theory, $h—y$ must be written for y.

The right arch equations used herein can be derived from the skew arch equations by making $\epsilon = 0$ and writing $h—y$ for y, or they can be derived as follows.

For Vertical Loading.—From Fig. 28:

$$T^v_x = \frac{\sum_A^B M y \frac{s}{I}}{\sum_A^B y^2 \frac{s}{I}} = \left[\frac{\sum_A^C \left(\frac{L-x'}{2L}\right)(L+x) y \frac{s}{I}}{\sum_A^C y^2 \frac{s}{I} + \sum_C^B y^2 \frac{s}{I}} + \frac{\sum_C^B \left(\frac{L-x'}{2L}\right)(L-x) y \frac{s}{I}}{\sum_A^C y^2 \frac{s}{I} + \sum_C^B y^2 \frac{s}{I}} - \frac{\sum_A^B (x-x') y \frac{s}{I}}{\sum_A^C + \frac{2s}{I} + \sum_C^B y^2 \frac{s}{I}} \right] P$$

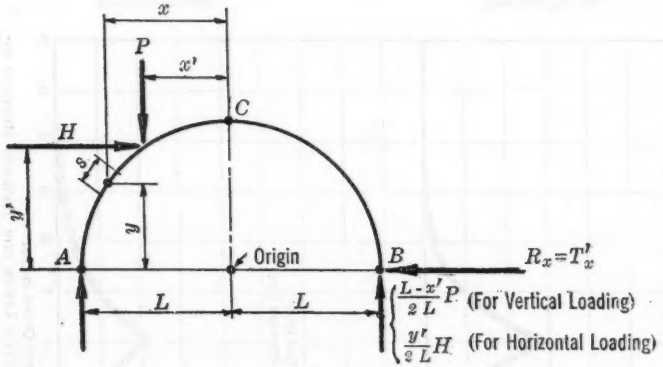


Fig. 28.

Since the arch is symmetrical, $\sum_A^C = \sum_B^C$:

$$T^v_x = \frac{(L-x')\sum_A^C y \frac{s}{I} - \sum_A^P x y \frac{s}{I} + x' \sum_A^P y \frac{s}{I}}{2\sum_A^C y^2 \frac{s}{I}} P$$

$$= \frac{a_v}{2\sum_A^C y^2 \frac{s}{I}} P \qquad (71)$$

TABLE 9.—CONSTANTS.

Point.	x.	y.	$\tan\phi$.	$\cos\phi$.	$\sin\phi$.	t.	I.	$\frac{1}{I}$.	$\frac{x}{I}$.	$\frac{x^2}{I}$.	$\frac{y}{I}$.	$\frac{y^2}{I}$.	$\frac{xy}{I}$.	$\frac{\sin\phi}{I}$.	$\frac{\sin^2\phi}{I}$.	$\frac{\sin\phi\cos\phi}{I}$.	$\frac{x\sin\phi\cos\phi}{I}$.	$\sum\frac{P_A}{I}\sin\phi\cos\phi$.	$\sum\frac{P_A}{I}x\sin\phi\cos\phi$.	$\sum\frac{P_R}{I}$.	$\sum\frac{P_R}{I}x$.	$\sum\frac{P_R}{I}x^2$.	$\sum\frac{P_A}{I}y$.	$\sum\frac{P_A}{I}y^2$.	$\sum\frac{P_A}{I}xy$.
1	26.2	1.6	∞	0	1	4.8	9.2			75.0	0.17	0.3	4.6	0.1087	0.1087	0	0	0	0			75	0.17	0.3	4.6
2	26.3	4.9	∞	0	1	2.7	1.64			422.0	2.99	14.6	78.6	0.610	0.610	0	0	0	0			497	3.16	14.9	83.2
3	26.5	8.2	∞	0	1	3.0	2.25			312.0	3.64	29.9	96.5	0.445	0.445	0	0	0	0			809	6.80	44.8	179.7
4	26.7	11.5	∞	0	1	3.31	3.02			236.0	3.81	43.8	101.7	0.331	0.331	0	0	0	0			1 045	10.61	88.6	281.4
5	26.2	14.7	0.423	0.921	0.390	3.54	3.70			186.0	3.97	58.4	104.0	0.1065	0.041	0.0971	2.54	0.0971	2.54			1 231	14.58	147.0	385.4
6	23.3	15.8	0.357	0.941	0.337	3.20	2.73		8.6		5.81	91.8	135.4	0.1247	0.042	0.117	2.73	0.214	5.27	0.37	8.6		20.39	238.8	620.8
7	20.2	16.5	0.324	0.951	0.309	2.50	1.30	0.37	15.5		12.69	209.2	256.4	0 288	0.0785	0.226	4.56	0.440	9.83	1.14	24.1		33.08	448.0	777.2
8	17.2	17.2	0.278	0.963	0.268	1.94	0.61	1.64	28.3		28.24	485.5	485.8	0.439	0.118	0.423	7.29	0.868	17.12	2.78	52.4		61.82	933.5	1 263.0
9	14.1	17.7	0.250	0.970	0.243	1.57	0.322	3.12	43.8		55.0	973.5	775.0	0.759	0.184	0.736	10.35	1.599	27.47	5.90	96.2		116.32	1 907	2 038.0
10	11.0	18.2	0.206	0.979	0.202	1.33	0.196	5.10	56.1		92.9	1 691.0	1 022.0	1.080	0.208	1.010	11.10	2.609	38.57	11.00	152.3		209.22	3 586	3 060
11	7.8	18.7	0.129	0.992	0.129	1.25	0.163	6.10	47.8		114.7	2 145.0	895.0	0.786	0.101	0.780	6.08	3.389	44.65	17.10	200.1		323.92	5 743	3 955
12	4.7	18.9	0.107	0.994	0.107	1.25	0.163	6.10	28.8		116.0	2 192.0	545.0	0.652	0.069	0.648	3.04	4.037	47.69	23.20	228.9		439.92	7 995	4 500
13	1.6	19.1	0	1.000	0	1.25	0.163	6.10	9.9		117.2	2 239.0	187.0	0	0	0	0	4.037	47.69	29.30	238.8		557.12	10 174	4 687

TABLE 10.—SOLUTION FOR T_x, T_y, T_z, FOR UNIT VERTICAL LOADING.

Load point.	x'.	$(L-x')$.	$(L-x')\sum_A^C \frac{y}{I}$.	$x'\sum_A^P \frac{y}{I}$.	a_v.	Crown thrust = $T_x = T'_x$.	$\frac{T_y = \frac{L-x'}{2L}}{}$.	$T_z = \epsilon T_x$.
5–6	24.8	1.4	780	362	757	0.037	0.027	0.0187
6–7	21.8	4.4	2 451	445	2 375	0.117	0.084	0.0596
7–8	18.7	7.5	4 178	620	4 021	0.198	0.143	0.1010
8–9	15.6	10.6	5 906	956	5 599	0.275	0.202	0.140
9–10	12.5	13.7	7 633	1 454	7 049	0.346	0.262	0.176
10–11	9.4	16.8	9 360	1 968	8 268	0.406	0.321	0.207
11–12	6.3	19.9	11 090	2 041	9 176	0.451	0.380	0.230
12–13	3.2	23.0	12 810	1 408	9 718	0.478	0.440	0.244
Crown	0	26.2	14 600	0	9 913	0.487	0.500	0.248

TABLE 11.—SOLUTION FOR T_x, T_y, T_z, FOR UNIT HORIZONTAL LOADING.

Load point.	y'.	$y'\sum_A^C \frac{y}{I}$.	$y'\sum_A^H \frac{y}{I}$.	a_h.	$T_x = T'_x$.	$\frac{T_y = \frac{y'}{2L}}{}$.	$T_z = \epsilon T_x$.
1–2	3.3	1 838	0.6	1 838	0.090	0.063	0.046
2–3	6.6	3 677	21.0	3 671	0.180	0.126	0.092
3–4	9.9	5 515	67.0	5 493	0.270	0.189	0.138
4–5	13.2	7 354	140.0	7 303	0.360	0.252	0.184
5–6	15.4	8 580	225.0	8 502	0.418	0.294	0.213

TABLE 12.—SOLUTION FOR M_x FOR VERTICAL LOADING.

Load point.	x'.	$\sum_B^P \frac{x}{I}$.	$\sum_B^P \frac{1}{I}$.	$x'\sum_B^P \frac{1}{I}$.	$\sum_B^P \frac{1}{I}\left(\frac{\sum_B^P \frac{x}{I}}{\sum_B^P \frac{1}{I}} - x'\right)$.	M_x.
6–7	21.8	8.6	0.37	8.07	+ 0.5	0.0
7–8	18.7	24.1	1.14	21.3	+ 2.8	0.024
8–9	15.6	52.4	2.78	43.3	+ 9.1	0.079
9–10	12.5	96.2	5.90	73.8	+ 22.4	0.195
10–11	9.4	152.3	11.00	103.4	+ 48.9	0.425
11–12	6.3	200.1	17.10	108.0	+ 92.1	0.800
12–13	3.2	228.9	23.20	74.2	+154.7	1.35
Crown	0	238.8	29.30	0	+238.8	2.08

TABLE 13.—SOLUTION FOR M_y FOR VERTICAL LOADING.

$\epsilon\sum_A^P \frac{x \sin\phi\cos\phi}{I}$.	$-\epsilon x'\sum_A^P \frac{\sin\phi\cos\phi}{I}$.	$-\epsilon\left(\sum_A^C \frac{x\sin\phi\cos\phi}{I}\right)T_x$.	$-\sum_A^C \frac{\sin^2\phi}{I}$.	M_y.
2.69	— 2.38	— 2.04	—2.329	0.814
5.02	— 4.20	— 3.48	—2.329	1.320
8.74	— 6.88	— 4.91	—2.329	1.71
14.00	—10.20	— 6.36	—2.329	1.91
19.65	—12.5	— 7.80	—2.329	1.82
22.8	—10.80	— 9.25	—2.329	1.40
24.3	— 6.6	—10.70	—2.329	0.795
24.3	0	—12.15	—2.329	0

TABLE 14.—TEMPERATURE STRESSES.

$E\,c\,t\,L = 288\,000 \times 0.0000065 \times 100 \times 26.25 = 4920$

$$T'_x = \frac{E\,c\,t\,L}{\sum y^2 \frac{s}{I}} = \frac{4920}{10170 \times 3.1} = 0.157 \text{ kips}$$

$T_x = T'_x \sec.^2\theta = 0.157 \times 1.26 = 0.198$ kips

$$T_z = \epsilon\left[T_x + \frac{1.34\,E\,c\,t\,L}{\sum_A^B x^2 \frac{s}{I}}\right] = 0.51\left(0.198 + \frac{1.34 \times 4920}{1231 \times 3.1}\right) = 0.983 \text{ kips}$$

For Horizontal Loading.—

$$T_x = \frac{y' \sum_A^C y \frac{s}{I} + \sum_A^H y^2 \frac{s}{I} - y' \sum_A^H y \frac{s}{I}}{2 \sum_A^C y^2 \frac{s}{I}} H = \frac{a_h}{2 \sum_A^C y^2 \frac{s}{I}} H \ldots (72)$$

For Temperature Rise.—

$$T_x = \frac{E c t L}{\sum_A^C y^2 \frac{s}{I}} \ldots (73)$$

Calculations for Influence Lines (See Fig. 29).

The following numerical example illustrates the use of the approximate theory and the saving of numerical work is apparent from a comparison of Tables 1 to 6, inclusive, and Tables 9 to 14, inclusive.

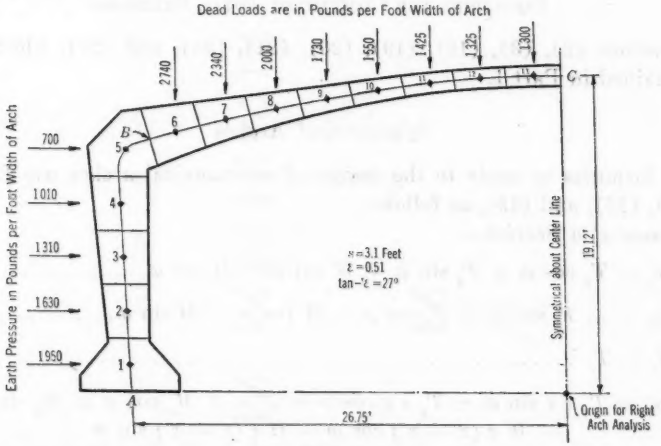


Fig. 29.

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APPENDIX I

In this section of the paper all final equations necessary for the design of skew arches are grouped together. It will be noted that the equation numbers are discontinuous. The purpose has been to arrange the formulas in such a way that any one using them may turn back to the text for an explanation or derivation. For example, Equations (16) follow Equations (3) in Appendix I. Both these sets are essential in a design, whereas Equations (1), (2), and (4) to (15), inclusive, etc., in the main paper, are necessary only for the derivation of the final design equations. The number of the equations in Appendix I serve to identify their normal position in the paper, where the derivations are given.

Equations for Computing Total Stresses

Equations (2), (3), (16), (19), (20), (21), (25), and (27), which follow, are explained in Part I.

Symmetrical Arches

For formulas to apply to the design of symmetrical arches use Equations (2), (3), (16), and (19), as follows:

Stresses at a Section.—

$$t_v = T_x \cos\phi \mp T_y \sin\phi + W \sin\phi - H \cos\phi \quad \ldots\ldots (2a)$$

$$t_u = \pm T_x \sin\phi + T_y \cos\phi - W \cos\phi - H \sin\phi \quad \ldots\ldots (2b)$$

$$t_z = T_z \quad \ldots\ldots (2c)$$

$$m_v = T_x\, \varepsilon\, x \sin\phi \pm T_y\, \varepsilon\, x \cos\phi - T_z\, u + M_x \cos\phi \mp M_y \sin\phi - W\, \varepsilon\, (x - x') \cos\phi - H\, \varepsilon\, (x - x') \sin\phi \quad \ldots\ldots (2d)$$

$$m_u = \mp T_x\, \varepsilon\, x \cos\phi + T_y\, \varepsilon\, x \sin\phi \pm T_z\, v \pm M_x \sin\phi + M_y \cos\phi - W\, \varepsilon\, (x - x') \sin\phi + H\, \varepsilon\, (x - x') \cos\phi \quad \ldots\ldots (2e)$$

$$m_z = T_x\, y \pm T_y\, x + M_z - W\, (x - x') - H\, (y - y') \quad \ldots\ldots (2f)$$

The upper signs in Equations (2) apply to the left half of the structure, and the lower signs apply to the right half. The loads, W and H, apply only to the left half of the arch.

Co-Ordinates of Crown Section with Respect to Any Point, P.—

$$u = x \sin\phi - y \cos\phi \quad \ldots\ldots (3a)$$

$$v = x \cos\phi + y \sin\phi \quad \ldots\ldots (3b)$$

Final Equations for the Fixed Symmetrical Arch (Origin of Co-Ordinates Is at Crown Center).—

$$K_4\, T_x + K_8\, M_z = C_4 \quad \ldots\ldots (16a)$$

$$K_1\, T_x + K_2\, T_z + K_3\, M_x + K_4\, M_z = C_1 \quad \ldots\ldots (16b)$$

$$K_2\, T_x + K_5\, T_z + K_6\, M_x = C_2 \quad \ldots\ldots (16c)$$

$$K_3\,T_x + K_6\,T_z + K_7\,M_x = C_3 \quad \ldots\ldots\ldots\ldots (16d)$$

$$K_9\,T_y + K_3\,M_y = C_5 \quad \ldots\ldots\ldots\ldots (16e)$$

and,

$$K_3\,T_y + K_{10}\,M_y = C_6 \quad \ldots\ldots\ldots\ldots (16f)$$

in which, the coefficients are, for all loading and temperature stresses:

$$K_1 = \sum y^2 \frac{s}{I} + \varepsilon^2 \sum x^2 \sin^2 \phi\,\lambda \frac{s}{F}$$

$$K_2 = -\varepsilon \sum u\,x \sin \phi\,\lambda \frac{s}{F}$$

$$K_3 = +\varepsilon \sum x \sin \phi \cos \phi\,\lambda \frac{s}{F}$$

$$K_4 = +\sum y \frac{s}{I}$$

$$K_5 = +\sum u^2 \lambda \frac{s}{F}$$

$$K_6 = -\sum u \cos \phi\,\lambda \frac{s}{F}$$

$$K_7 = +\sum \cos^2 \phi\,\lambda \frac{s}{F}$$

$$K_8 = +\sum \frac{s}{I}$$

$$K_9 = -\sum x^2 \frac{s}{I} - \varepsilon^2 \sum x^2 \cos^2 \phi\,\lambda \frac{s}{F}$$

and,

$$K_{10} = -\sum \sin^2 \phi\,\lambda \frac{s}{F}$$

For a vertical load, W, at the point, $x = x'$, $z = \epsilon\,x'$:

$$C_1 = \frac{1}{2}\left[\sum_A^P x\,y \frac{s}{I} - x' \sum_A^P y \frac{s}{I} + \varepsilon^2 \sum_A^P x^2 \sin \phi \cos \phi\,\lambda \frac{s}{F} - \varepsilon^2\,x' \sum_A^P x \sin \phi \cos \phi\,\lambda \frac{s}{F}\right] W$$

$$C_2 = \frac{1}{2}\left[-\varepsilon \sum_A^P u\,x \cos \phi\,\lambda \frac{s}{F} + \varepsilon\,x' \sum_A^P u \cos \phi\,\lambda \frac{s}{F}\right] W$$

$$C_3 = \frac{1}{2}\left[\varepsilon \sum_A^P x \cos^2 \phi\,\lambda \frac{s}{F} - \varepsilon\,x' \sum_A^P \cos^2 \phi\,\lambda \frac{s}{F}\right] W$$

$$C_4 = \frac{1}{2}\left[\sum_A^P x \frac{s}{I} - x' \sum_A^P \frac{s}{I}\right] W$$

$$C_5 = \frac{1}{2}\left[-\sum_A^P x^2 \frac{s}{I} + x' \sum_A^P x \frac{s}{I} - \varepsilon^2 \sum_A^P x^2 \cos^2 \phi\,\lambda \frac{s}{F} + \varepsilon^2\,x' \sum_A^P x \cos^2 \phi\,\lambda \frac{s}{F}\right] W$$

and,

$$C_6 = \frac{1}{2}\left[\varepsilon \sum_A^P x \sin\phi\cos\phi\,\lambda\,\frac{s}{F} - \varepsilon\,x' \sum_A^P \sin\phi\cos\phi\,\lambda\,\frac{s}{F}\right] W$$

For a horizontal load, H, at the point, $y = y', z = \varepsilon\,x'$:

$$C_1 = \frac{1}{2}\left[\sum_A^P y^2\,\frac{s}{I} - y' \sum_A^P y\,\frac{s}{I} + \varepsilon^2 \sum_A^P x^2 \sin^2\phi\,\lambda\,\frac{s}{F} - \varepsilon^2\,x' \sum_A^P x \sin^2\phi\,\lambda\,\frac{s}{F}\right] H$$

$$C_2 = \frac{1}{2}\left[-\varepsilon \sum_A^P u\,x \sin\phi\,\lambda\,\frac{s}{F} + \varepsilon\,x' \sum_A^P u \sin\phi\,\lambda\,\frac{s}{F}\right] H$$

$$C_3 = \frac{1}{2}\left[\varepsilon \sum_A^P x \sin\phi\cos\phi\,\lambda\,\frac{s}{F} - \varepsilon\,x' \sum_A^P \sin\phi\cos\phi\,\lambda\,\frac{s}{F}\right] H$$

$$C_4 = \frac{1}{2}\left[\sum_A^P y\,\frac{s}{I} - y' \sum_A^P \frac{s}{I}\right] H$$

$$C_5 = \frac{1}{2}\left[-\sum_A^P x\,y\,\frac{s}{I} + y' \sum_A^P x\,\frac{s}{I} - \varepsilon^2 \sum_A^P x^2 \sin\phi\cos\phi\,\lambda\,\frac{s}{F} + \varepsilon^2\,x' \sum_A^P x \sin\phi\cos\phi\,\lambda\,\frac{s}{F}\right] H$$

and,

$$C_6 = \frac{1}{2}\left[\varepsilon \sum_A^P x \sin^2\phi\,\lambda\,\frac{s}{F} - \varepsilon\,x' \sum_A^P \sin^2\phi\,\lambda\,\frac{s}{F}\right] H$$

For temperature stresses ($t°$ rise):

$$C_1 = E\,c\,t\,L;\ C_2 = \varepsilon\,E\,c\,t\,L;\ \text{and, } C_3 = C_4 = C_5 = C_6 = 0$$

All the summations are from A to C (Fig. 5) unless otherwise noted.

Final Equations for the Two-Hinged Symmetrical Arch (Origin of Co-Ordinates at Crown Center).—As in the case of the fixed arch, the final equations are written in terms of the coefficients, K and C:

$$K_{11}\,T_x + K_2\,T_z + K_3\,M_x = C_{11} \quad \ldots\ldots (19a)$$

$$K_2\,T_x + K_5\,T_z + K_6\,M_x = C_2 \quad \ldots\ldots (19b)$$

$$K_3\,T_x + K_6\,T_z + K_7\,M_x = C_3 \quad \ldots\ldots (19c)$$

$$M_z = C_{14} - h\,T_x \quad \ldots\ldots (19d)$$

$$T_y = C_{15} \quad \ldots\ldots (19e)$$

$$K_{10}\,M_y = C_6 - K_3\,T_x \quad \ldots\ldots (19f)$$

All the coefficients given in Equations (19) have the same values as those in Equations (16), with the following additional values:

For all loading and temperature stresses:

$$K_{11} = \sum y^2\,\frac{s}{I} - 2\,h \sum y\,\frac{s}{I} + h^2 \sum \frac{s}{I} + \varepsilon^2 \sum x^2 \sin^2\phi\,\frac{\lambda\,s}{F}$$

For a vertical load, W, at the point $x = x'$, $z = \varepsilon x'$:

$$C_{11} = \frac{1}{2}\left[(L - x')\left(-\sum y\frac{s}{I} + h\sum\frac{s}{I}\right) + \sum_A^P x\,y\frac{s}{I} - x'\sum_A^P y\frac{s}{I} - h\sum_A^P x\frac{s}{I} + h\,x'\sum_A^P\frac{s}{I} + \varepsilon^2\sum_A^P x^2\sin\phi\cos\phi\frac{\lambda\,s}{F} - \varepsilon^2\,x'\sum_A^P x\sin\phi\cos\phi\frac{\lambda\,s}{F}\right]W$$

$$C_{14} = \frac{1}{2}(L - x')\,W$$

$$C_{15} = \frac{1}{2}\left(1 - \frac{x'}{L}\right)W$$

For a horizontal load, H, at the point, $y = y'$, $z = \varepsilon x'$:

$$C_{11} = \frac{1}{2}\left[(h - y')\left(-\sum y\frac{s}{I} + h\sum\frac{s}{I}\right) + \sum_A^P y^2\frac{s}{I} - y'\sum_A^P y\frac{s}{I} - h\sum_A^P y\frac{s}{I} + h\,y'\sum_A^P\frac{s}{I} + \varepsilon^2\sum_A^P x^2\sin^2\phi\frac{\lambda\,s}{F} - \varepsilon^2\,x'\sum_A^P x\sin^2\phi\frac{\lambda\,s}{F}\right]H$$

$$C_{14} = \frac{1}{2}(h - y')\,H$$

$$C_{15} = \frac{1}{2}\left(\frac{h - y'}{L}\right)H$$

Unsymmetrical Arches

For formulas to apply to the design of unsymmetrical arches, use Equations (20), (21), (25), and (27), as follows:

Stresses at a Section.—

$$t_v = -R_x\cos\phi - R_y\sin\phi + W\sin\phi - H\cos\phi \quad\ldots\ldots(20a)$$

$$t_u = -R_x\sin\phi + R_y\cos\phi - W\cos\phi - H\sin\phi \quad\ldots\ldots(20b)$$

$$t_z = R_z \quad\ldots\ldots(20c)$$

$$m_u = R_x\,\varepsilon\,x\cos\phi + R_y\,\varepsilon\,x\sin\phi + R_z\,v' - N_x\sin\phi + N_y\cos\phi - W\,\varepsilon\,(x - x')\sin\phi + H\,\varepsilon\,(x - x')\cos\phi \quad\ldots\ldots(20d)$$

$$m_v = -R_x\,\varepsilon\,x\sin\phi + R_y\,\varepsilon\,x\cos\phi - R_z\,u' - N_x\cos\phi - N_y\sin\phi - W\,\varepsilon\,(x - x')\cos\phi - H\,\varepsilon\,(x - x')\sin\phi \quad\ldots\ldots(20e)$$

$$m_z = R_x\,y + R_y\,x + N_z - W\,(x - x') + H\,(y - y') \quad\ldots\ldots(20f)$$

Co-Ordinates of the Right Abutment with Respect to Any Point, P.—

$$u' = x\sin\phi + y\cos\phi \quad\ldots\ldots(21a)$$

$$v' = x\cos\phi - y\sin\phi \quad\ldots\ldots(21b)$$

Final Equations for Unsymmetrical Fixed Arch (Origin of Co-Ordinates Is at Center of Right Abutment).—Here, again, the final equations are written in terms of the coefficients, K and C, for brevity:

$$-K_1 R_x + K_2 R_y + K_3 R_z + K_4 N_x - K_5 N_y = C_1. \quad (25a)$$

$$K_6 R_x - K_7 R_y + K_8 R_z + K_5 N_x + K_9 N_y = C_2. \quad (25b)$$

$$K_{10} R_x + K_{11} R_y + K_{12} N_z = C_3. \quad (25c)$$

$$K_{13} R_x + K_{14} R_y + K_{15} R_z + K_7 N_x + K_6 N_y + K_{10} N_z = C_4. \quad (25d)$$

$$K_{14} R_x + K_{17} R_y - K_{18} R_z - K_2 N_x - K_7 N_y + K_{11} N_z = C_5. \quad (25e)$$

$$-K_{15} R_x + K_{18} R_y + K_{19} R_z + K_3 N_x - K_8 N_y = C_6. \quad (25f)$$

in which, the coefficients for all loading and temperature stresses are:

$$K_1 = \varepsilon \sum x \sin\phi \cos\phi \frac{\lambda s}{F} \qquad K_{11} = \sum x \frac{s}{I}$$

$$K_2 = \varepsilon \sum x \cos^2\phi \frac{\lambda s}{F} \qquad K_{12} = \sum \frac{s}{I}$$

$$K_3 = -\sum u' \cos\phi \frac{\lambda s}{F} \qquad K_{13} = \sum y^2 \frac{s}{I} + \varepsilon^2 \sum x^2 \sin^2\phi \frac{\lambda s}{F}$$

$$K_4 = -\sum \cos^2\phi \frac{\lambda s}{F} \qquad K_{14} = \sum x y \frac{s}{I} - \varepsilon^2 \sum x^2 \sin\phi \cos\phi \frac{\lambda s}{F}$$

$$K_5 = \sum \sin\phi \cos\phi \frac{\lambda s}{F} \qquad K_{15} = \varepsilon \sum u' x \sin\phi \frac{\lambda s}{F}$$

$$K_6 = \varepsilon \sum x \sin^2\phi \frac{\lambda s}{F} \qquad K_{16} = \varepsilon \sum x \sin^2\phi \frac{\lambda s}{F}$$

$$K_7 = \varepsilon \sum x \sin\phi \cos\phi \frac{\lambda s}{F} \qquad K_{17} = \sum x^2 \frac{s}{I} + \varepsilon^2 \sum x^2 \cos^2\phi \frac{\lambda s}{F}$$

$$K_8 = \sum u' \sin\phi \frac{\lambda s}{F} \qquad K_{18} = \varepsilon \sum u' x \cos\phi \frac{\lambda s}{F}$$

$$K_9 = \sum \sin^2\phi \frac{\lambda s}{F} \qquad K_{19} = -\sum u'^2 \frac{\lambda s}{F}$$

$$K_{10} = \sum y \frac{s}{I}$$

For vertical loading:

$$C_1 = \varepsilon \left[\sum_A^P x \cos^2\phi \frac{\lambda s}{F} - x' \sum_A^P \cos^2\phi \frac{\lambda s}{F} \right] W$$

$$C_2 = \varepsilon \left[-\sum_A^P x \sin\phi \cos\phi \frac{\lambda s}{F} + x' \sum_A^P \sin\phi \cos\phi \frac{\lambda s}{F} \right] W$$

$$C_3 = \left[\sum_A^P x \frac{s}{I} - x' \sum_A^P \frac{s}{I} \right] W$$

$$C_4 = \left[\sum_A^P x y \frac{s}{I} - x' \sum_A^P y \frac{s}{I} + \varepsilon^2 \left(-\sum_A^P x^2 \sin\phi \cos\phi \frac{\lambda s}{F} + x' \sum_A^P x \sin\phi \cos\phi \frac{\lambda s}{F} \right) \right] W$$

$$C_5 = \left[\sum_A^P x^2 \frac{s}{I} - x' \sum_A^P x \frac{s}{I} + \varepsilon^2 \left(\sum_A^P x^2 \cos^2 \phi \frac{\lambda s}{F} - x' \sum_A^P x \cos^2 \phi \frac{\lambda s}{F}\right)\right] W$$

$$C_6 = \varepsilon \left[\sum_A^P u' x \cos \phi \frac{\lambda s}{F} - x' \sum_A^P u' \cos \phi \frac{\lambda s}{F}\right] W$$

For horizontal loading:

$$C_1 = \varepsilon \left[\sum_A^H x \sin \phi \cos \phi \frac{\lambda s}{F} - x' \sum_A^H \sin \phi \cos \phi \frac{\lambda s}{F}\right] H$$

$$C_2 = \varepsilon \left[-\sum_A^H x \sin^2 \phi \frac{\lambda s}{F} + x' \sum_A^H \sin^2 \phi \frac{\lambda s}{F}\right] H$$

$$C_3 = \left[-\sum_A^H y \frac{s}{I} + y' \sum_A^H \frac{s}{I}\right] H$$

$$C_4 = \left[-\sum_A^H y^2 \frac{s}{I} + y' \sum_A^H y \frac{s}{I} + \varepsilon^2 \left(-\sum_A^H x^2 \sin^2 \phi \frac{\lambda s}{F} + x' \sum_A^H x \sin^2 \phi \frac{\lambda s}{F}\right)\right] H$$

$$C_5 = \left[-\sum_A^H x y \frac{s}{I} + y' \sum_A^H x \frac{s}{I} + \varepsilon^2 \left(\sum_A^H x^2 \sin \phi \cos \phi \frac{\lambda s}{F} - x' \sum_A^H x \sin \phi \cos \phi \frac{\lambda s}{F}\right)\right] H$$

$$C_6 = \varepsilon \left[\sum_A^H u' x \sin \phi \frac{\lambda s}{F} - x' \sum_A^H u' \sin \phi \frac{\lambda s}{F}\right] H$$

For temperature stresses (t ° rise):

$$C_1 = C_2 = C_3 = 0$$

$$C_4 = -2 E c t L$$

$$C_5 = E c t a$$

$$C_6 = -2 \varepsilon E c t L$$

All the summations are from A to B (Fig. 9) unless otherwise noted.

Final Equations for Unsymmetrical Two-Hinged Arch (Origin of Co-Ordinates Is at Center of Right Abutment).—

$$-K_1 R_x + K_2 R_y + K_3 R_z + K_4 N_x - K_5 N_y = C_1 \quad \ldots\ldots (27a)$$

$$K_6 R_x - K_7 R_y + K_8 R_z + K_5 N_x + K_9 N_y = C_2 \quad \ldots\ldots (27b)$$

$$K_{13} R_x + K_{14} R_y + K_{15} R_z + K_7 N_x + K_6 N_y - \left(\frac{a}{2L}\right)(K_{14} R_x + K_{17} R_y - K_{18} R_z - K_2 N_x - K_7 N_y) = C_4 - \left(\frac{a}{2L}\right) C_5 \quad \ldots (27c)$$

$$-K_{15} R_x + K_{18} R_y + K_{19} R_z + K_3 N_x - K_8 N_y = C_6 \quad \ldots\ldots (27d)$$

$$R_y + \left(\frac{a}{2L}\right) R_x = C_7 \quad \ldots\ldots (27e)$$

$$N_z = 0 \quad \ldots\ldots (27f)$$

All the coefficients are the same as in Equation (26) for the fixed arch except C_7, which varies under different conditions as follows:

For vertical loading, W,

$$C_7 = \left(1 - \frac{x'}{2\ L}\right) W$$

For horizontal loading, H,

$$C_7 = \left(\frac{y'}{2\ L}\right) H$$

For temperature stresses,

$$C_7 = 0$$

Proportioning Sections

All the following equations are fully explained in Part II.

Relations Between Angles

$$\varepsilon' = \varepsilon \cos \phi \quad \ldots\ldots (28)$$

$$\tan \phi' = \tan \phi \cos \theta \quad \ldots\ldots (29)$$

Equations for Stresses per Unit Width of Arch

$$t_s = \frac{3}{2}\left(\frac{t_z - \varepsilon'\ t_v}{b}\right)\left(1 - \frac{4\ z^2}{b^2}\right) + \varepsilon' \frac{t_v}{b} + 12\ \varepsilon' \frac{m_u z}{b^3} \quad \ldots\ldots (35)$$

$$t'_n = \varepsilon'\ t_s \quad \ldots\ldots (36)$$

$$m_s = \frac{\varepsilon'\ m_z}{b} \text{ (at the face of bridge)} \quad \ldots\ldots (43)$$

$$m_s = \frac{m_v + \varepsilon'\ m_z}{2\ b} \text{ (at any other point of the section)} \quad \ldots\ldots (44)$$

$$m' = \frac{\varepsilon'^2\ m_z}{b} \text{ (at the face of bridge)} \quad \ldots\ldots (48)$$

$$m' = \frac{\varepsilon'\ m_v}{b} \text{ (at any other point of the section)} \quad \ldots\ldots (49)$$

$$V' = \frac{\varepsilon'\ t_u}{b} - \frac{8}{5}\ \varepsilon' \frac{(m_v - \varepsilon'\ m_z)}{b\ t} \text{ (at the face of bridge)} \quad \ldots\ldots (51)$$

$$V' = \frac{\varepsilon'\ t_u}{b} \text{ (at any other point of the section)} \quad \ldots\ldots (52)$$

Transformation Formulas

Equations (53a) and (53b) are general formulas:

$$\left.\begin{aligned} M &= m \sin^2 \psi + m' \cos^2 \psi + 2\ m_s \sin \psi \cos \psi \\ N &= t_n \sin^2 \psi + t'_n \cos^2 \psi + 2\ t_s \sin \psi \cos \psi \end{aligned}\right\} \quad \ldots\ldots (53a)$$

$$\left.\begin{aligned} M_s &= (m' - m) \sin \psi \cos \psi + m_s (\sin^2 \psi - \cos^2 \psi) \\ S &= (t'_n - t_n) \sin \psi \cos \psi + t_s (\sin^2 \psi - \cos^2 \psi) \end{aligned}\right\} \quad \ldots\ldots (53b)$$

They may be adapted to special cases in the following ways:

For Principal Moment Planes.—

$$\left.\begin{aligned} \tan 2\psi &= \frac{2\,m_s}{m' - m} \\ M &= \frac{m' + m}{2} \pm \frac{1}{2}\sqrt{(m' - m)^2 + 4\,m_s^2} \end{aligned}\right\} \quad \ldots\ldots\ldots (53c)$$

$$\left.\begin{aligned} \psi &= 90 - \theta' \\ M &= m \sec^2 \theta' \end{aligned}\right\} \text{ (at the face of bridge)} \quad \ldots\ldots\ldots\ldots\ldots (53d)$$

For Principal Thrust Planes.—

$$\left.\begin{aligned} \tan 2\psi &= \frac{2\,t_s}{t'_n - t_n} \\ N &= \frac{t'_n + t_n}{2} \pm \frac{1}{2}\sqrt{(t'_n - t_n)^2 + 4\,t_s^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (53e)$$

$$\left.\begin{aligned} \psi &= 90 - \theta' \\ N &= t_n \sec^2 \theta' \end{aligned}\right\} \text{ (at the face of bridge)} \quad \ldots\ldots\ldots\ldots\ldots (53f)$$

For Maximum and Minimum Torsion Moment.—

$$\left.\begin{aligned} \tan 2\psi &= \frac{m - m'}{2\,m_s} \\ M_s &= \pm \frac{1}{2}\sqrt{(m' - m)^2 + 4\,m_s^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (53g)$$

For Maximum and Minimum Shearing Thrust.—

$$\left.\begin{aligned} \tan 2\psi &= \frac{t_n - t'_n}{2\,t_s} \\ S &= \pm \frac{1}{2}\sqrt{(t'_n - t_n)^2 + 4\,t_s^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (53h)$$

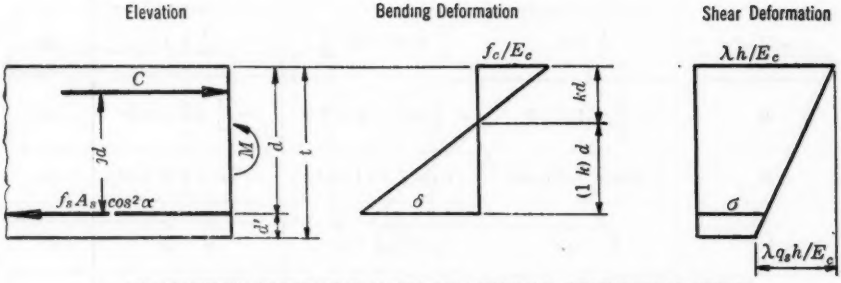
Final Design Formulas (Equations (64))

CASE I.—Simple Bending; Steel in One Direction, in the Intrados Face Only (See Fig. 30).

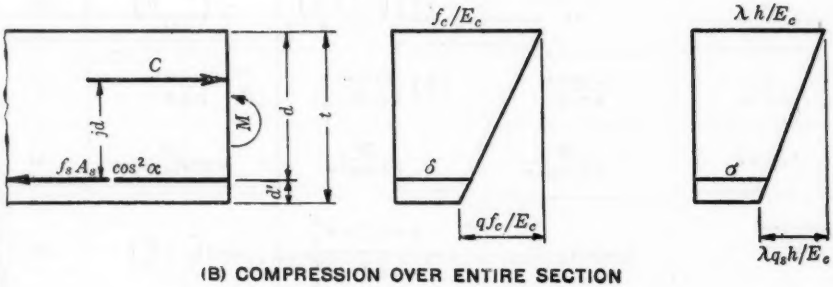
	(*A*) Tension over part of section containing steel.	(*B*) Compression over entire section.	(*C*) Tension over part of section not containing steel.	Equations (64).
$f_s p \cos^2 \alpha =$	$\frac{1}{2} f_c k$	$\frac{1}{2} f_c (2-q) \frac{t}{d}$	$\frac{1}{2} f_c k$	(*a*)
$M =$	$\frac{1}{2} f_c k j b d^2$	$\frac{1}{2} f_c (2-q) j b t^2$	$\frac{1}{2} f_c k j b d^2$	(*b*)
$M =$	$f_s p j b d^2 \cos^2 \alpha$	$f_s p j b d t \cos^2 \alpha$	$f_s p j b d^2 \cos^2 \alpha$	(*c*)
$j =$	$1 - \frac{k}{3}$	$\frac{d}{t} - \frac{3-2q}{6-3q}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a =$	$\frac{1-k}{k}$	$-\left(1 - q \frac{d}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*e*)
$c =$	$\frac{1-k}{k^2}$	$-\left(\frac{d}{t}\right)\left(\frac{1-q\frac{d}{t}}{2-q}\right)$	$-\left(\frac{k-\frac{d'}{d}}{k^2}\right)$	(*f*)
$\frac{1}{2pn} =$	$\frac{f_s \cos^2 \alpha}{n f_c k}$	$\left(\frac{d}{t}\right)\frac{f_s \cos^2 \alpha}{n f_c (2-q)}$	$\frac{f_s \cos^2 \alpha}{n f_c k}$	(*g*)
$f_s p b d =$	$\frac{M}{j d \cos^2 \alpha}$	$\frac{M}{j t \cos^2 \alpha}$	$\frac{M}{j d \cos^2 \alpha}$	(*h*)
	$k = 3(1-j)$	$q = \frac{3 + 6j - 6\frac{d}{t}}{2 + 3j - 3\frac{d}{t}}$	$k = 3\left(j + \frac{d'}{d}\right)$	(*i*)

$f_s p \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) \frac{t}{d}$	(*a'*)
$M_s =$	$\frac{1}{2} h (2 - j_s) b j_s t^2$	(*b'*)
$M_s =$	$f_s p j_s b d t \sin \alpha \cos \alpha$	(*c'*)
j_s* $=$	$\frac{d}{t} - \frac{3-2q_s}{6-3q_s}$	(*d'*)
$a_s =$	$-\left(1 - q_s \frac{d}{t}\right)$	(*e'*)
$c_s =$	$-\left(\frac{d}{t}\right)\left(\frac{1-q_s\frac{d}{t}}{2-q_s}\right)$	(*f'*)
$\frac{1}{2pn} =$	$\left(\frac{d}{t}\right)\frac{f_s \sin \alpha \cos \alpha}{n h (2-q_s)}$	(*g'*)
$f_s p b d =$	$\frac{M_s}{j_s t \sin \alpha \cos \alpha}$	(*h'*)
q_s* $=$	$\frac{3 + 6 j_s - 6 \frac{d}{t}}{2 + 3 j_s - 3 \frac{d}{t}}$	(*i'*)
$\frac{f_s}{n} =$	$\Sigma\Sigma\, f_c a \cos^2 \alpha + \lambda h a_s \sin \alpha \cos \alpha$	(*j*)
$\frac{1}{2pn} =$	$\Sigma\Sigma\, c \cos^4 \alpha + \lambda c_s \sin^2 \alpha \cos^2 \alpha$	(*k*)

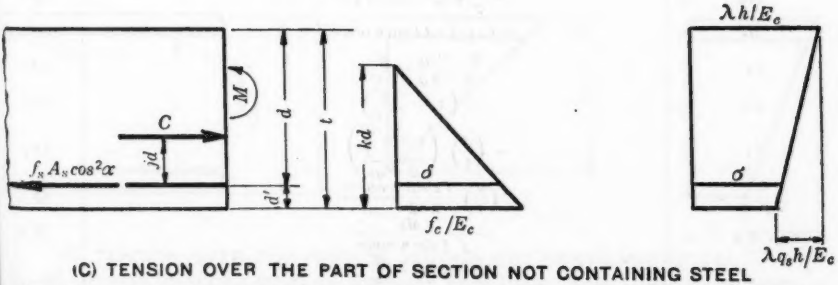
*If in any of the equations, $q_s = 2$, and $j_s = \infty$, that equation becomes indeterminate. By differentiation, it can be reduced to a determinate value: $\pm \frac{6 M_s}{b t^2} = h$, which means that the resultant is a couple.



(A) TENSION OVER THE PART OF SECTION CONTAINING STEEL



(B) COMPRESSION OVER ENTIRE SECTION

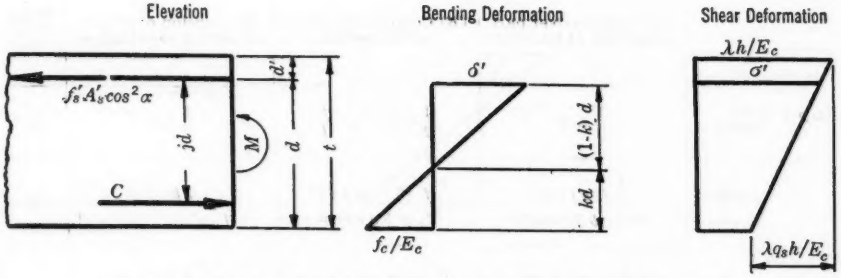


(C) TENSION OVER THE PART OF SECTION NOT CONTAINING STEEL

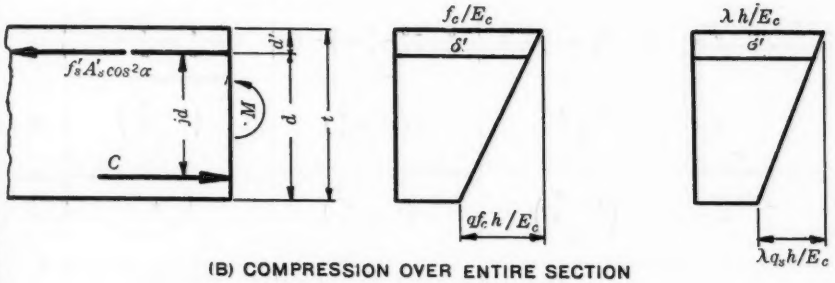
FIG. 30.—SIMPLE BENDING; STEEL IN INTRADOS FACE ONLY. (SEE CASE I FOR STEEL IN ONE DIRECTION AND CASE VII FOR STEEL IN TWO DIRECTIONS.)

CASE II.—Simple Bending: Steel in One Direction in Extrados Face Only (See Fig. 31).

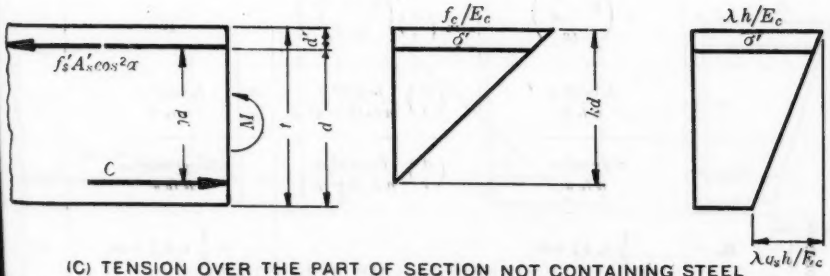
	(*A*) Tension over part of section containing steel.	(*B*) Compression over entire section.	(*C*) Tension over part of section not containing steel.	Equations (64).
$f'_s p' \cos^2 \alpha =$	$\frac{1}{2} f_c k$	$\frac{1}{2} f_c (2-q) \frac{t}{d}$	$\frac{1}{2} f_c k$	(*a*)
$M =$	$-\frac{1}{2} f_c k j' b d^2$	$-\frac{1}{2} f_c (2-q) j' b t^2$	$-\frac{1}{2} f_c k j' b d^2$	(*b*)
$M =$	$-f'_s p' j' b d^2 \cos^2 \alpha$	$-f'_s p' j' b d t \cos^2 \alpha$	$-f'_s p' j' b d^2 \cos^2 \alpha$	(*c*)
$j' =$	$\left(1-\frac{k}{3}\right)$	$\frac{3-2q}{6-3q} - \frac{d'}{t}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a' =$	$\frac{1-k}{k}$	$-\left(1-q\frac{d'}{t}\right)$	$-\left(\frac{k-\frac{d'}{d}}{k}\right)$	(*e*)
$c' =$	$\frac{1-k}{k^2}$	$-\left(\frac{d}{t}\right)\left(\frac{1-q\frac{d'}{t}}{2-q}\right)$	$-\left(\frac{k-\frac{d'}{d}}{k^2}\right)$	(*f*)
$\frac{1}{2p'n} =$	$\frac{f'_s \cos^2 \alpha}{n f_c k}$	$\left(\frac{d}{t}\right)\frac{f'_s \cos^2 \alpha}{n f_c (2-q)}$	$\frac{f'_s \cos^2 \alpha}{n f_c k}$	(*g*)
$f'_s p' b d =$	$-\frac{M}{j' d \cos^2 \alpha}$	$-\frac{M}{j' t \cos^2 \alpha}$	$-\frac{M}{j' d \cos^2 \alpha}$	(*h*)
	$k = 3(1-j')$	$q = \frac{3-6j'-6\frac{d'}{t}}{2-3j'-3\frac{d'}{t}}$	$k = 3\left(j'+\frac{d'}{d}\right)$	(*i*)
$f'_s p' \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2-q_s) \frac{t}{d}$			(*a'*)
$M_s =$	$-\frac{1}{2} h (2-q_s) b j'_s t^2$			(*b'*)
$M_s =$	$-f'_s p' j b d t \sin \alpha \cos \alpha$			(*c'*)
$j'_s =$	$\frac{3-2q_s}{6-3q} - \frac{d'}{t}$			(*d'*)
$a'_s =$	$-\left(1-q_s\frac{d'}{t}\right)$			(*e'*)
$c'_s =$	$-\left(\frac{d}{t}\right)\left(\frac{1-q\frac{d'}{t}}{2-q_s}\right)$			(*f'*)
$\frac{1}{2p'n} =$	$\left(\frac{d}{t}\right)\frac{f'_s \sin \alpha \cos \alpha}{h (2-q_s)}$			(*g'*)
$f'_s p' b d =$	$-\frac{M_s}{j' t \sin \alpha \cos \alpha}$			(*h'*)
$q_s =$	$\frac{3-6j'_s-6\frac{d'}{t}}{2-3j'_s-3\frac{d'}{t}}$			(*i'*)
$\frac{f'_s}{n} =$	$\sum\sum f_c a' \cos^2 \alpha + \lambda h a'_s \sin \alpha \cos \alpha$			(*j*)
$\frac{1}{2p'n} =$	$\sum\sum c' \cos^4 \alpha + \lambda c'_s \sin^2 \alpha \cos^2 \alpha$			(*k*)



(A) TENSION OVER THE PART OF SECTION CONTAINING STEEL



(B) COMPRESSION OVER ENTIRE SECTION

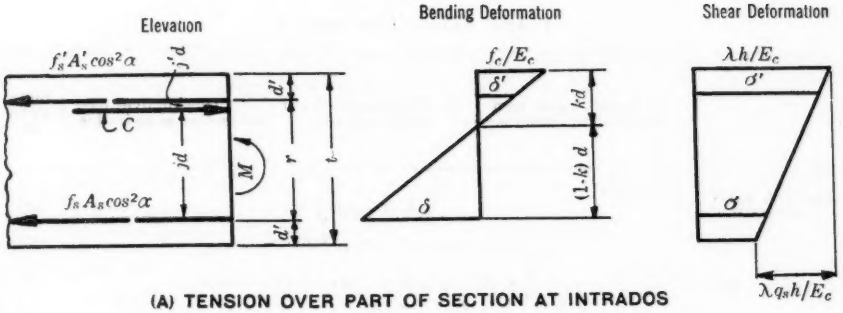


(C) TENSION OVER THE PART OF SECTION NOT CONTAINING STEEL

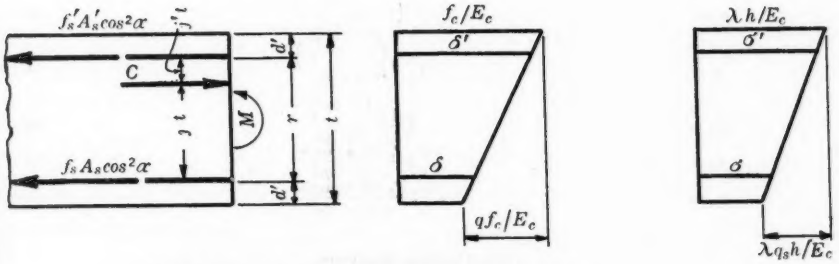
FIG. 31.—SIMPLE BENDING; STEEL IN EXTRADOS FACE ONLY. (SEE CASE II FOR STEEL IN ONE DIRECTION AND CASE VIII FOR STEEL IN TWO DIRECTIONS.)

CASE III.—Simple Bending; Steel in One Direction Only but in Both Faces (See Fig. 32).

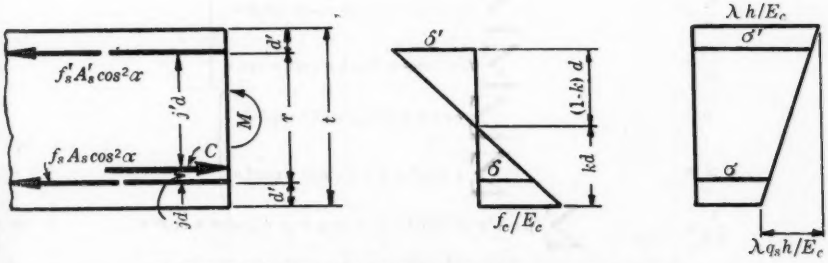
		(*A*) Tension over part of section at intrados	(*B*) Compression over entire section	(*C*) Tension over part of section at extrados	Equations (64)
$(f_s p + f'_s p') \cos^2 \alpha =$		$\frac{1}{2} f_c k$	$\frac{1}{2} f_c (2-q) \frac{t}{d}$	$\frac{1}{2} f_c k$	(*a*)
$M =$		$\frac{1}{2} f_c k j b d^2 - f'_s p' b d r \cos^2 \alpha$	$\frac{1}{2} f_c (2-q) j b t^2 - f'_s p' b d r \cos^2 \alpha$	$\frac{1}{2} f_c k j b d^2 - f'_s p' b d r \cos^2 \alpha$	(*b*)
$M =$		$-\frac{1}{2} f_c k j' b d^2 + f_s p b d r \cos^2 \alpha$	$-\frac{1}{2} f_c (2-q) j' b t^2 + f_s p b d r \cos^2 \alpha$	$-\frac{1}{2} f_c k j' b d^2 + f_s p b d r \cos^2 \alpha$	(*c*)
$j =$		$1 - \frac{k}{3} = \frac{r}{d} - j'$	$\frac{d}{t} - \frac{3-2q}{6-3q} = \frac{r}{t} - j'$	$\frac{k}{3} - \frac{d'}{d} = \left(\frac{r}{d} - j'\right)$	(*d*)
$j' =$		$\frac{k}{3} - \frac{d'}{d} = \frac{r}{d} - j$	$\frac{3-2q}{6-3q} - \frac{d'}{t} = \frac{r}{t} - j$	$1 - \frac{k}{3} = \frac{r}{d} - j$	(*e*)
$a =$		$\frac{1-k}{k}$	$-\left(1 - q\frac{d}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*f*)
$a' =$		$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	$-\left(1 - q\frac{d'}{t}\right)$	$\frac{1-k}{k}$	(*g*)
$c =$		$\frac{1-k}{k^2}$	$-\left(\frac{d}{t}\right)\left(\frac{1 - q\frac{d}{t}}{2-q}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k^2}\right)$	(*h*)
$c' =$		$-\left(\frac{k - \frac{d'}{d}}{k^2}\right)$	$-\left(\frac{d}{t}\right)\left(\frac{1 - q\frac{d'}{t}}{2-q}\right)$	$\frac{1-k}{k^2}$	(*i*)
$g =$		$\frac{f_s \cos^2 \alpha}{n f_c k}$	$\left(\frac{d}{t}\right) \frac{f_s \cos^2 \alpha}{n f_c (2-q)}$	$\frac{f_s \cos^2 \alpha}{n f_c k}$	(*j*)
$g' =$		$\frac{f'_s \cos^2 \alpha}{n f_c k}$	$\left(\frac{d}{t}\right) \frac{f'_s \cos^2 \alpha}{n f_c (2-q)}$	$\frac{f'_s \cos^2 \alpha}{n f_c k}$	(*k*)
For partial moments	$M_c =$	$\frac{1}{2} f_c k j b d^2$		$-\frac{1}{2} f_c k j' b d^2$	(*l*)
	$M - M_c =$	$f_s p_2 b d r \cos^2 \alpha$		$-f'_s p'_2 b r \cos^2 \alpha$	(*m*)
	$M - M_c =$	$-f'_s p' b d r \cos^2 \alpha$		$f_s p b d r \cos^2 \alpha$	(*n*)



(A) TENSION OVER PART OF SECTION AT INTRADOS



(B) COMPRESSION OVER ENTIRE SECTION



(C) TENSION OVER PART OF SECTION AT EXTRADOS

FIG. 32.—SIMPLE BENDING; STEEL IN BOTH FACES. (SEE CASE III FOR STEEL IN ONE DIRECTION AND CASE IX FOR STEEL IN TWO DIRECTIONS.)

CASE III.—(*Continued.*)

	(*A*) Tension over part of section at intrados	(*B*) Compression over entire section	(*C*) Tension over part of section at extrados	Equations (64)
$(f_s p + f_s' p') \sin \alpha \cos \alpha =$		$\frac{1}{2} h (2 - q_s) \frac{t}{d}$		(a')
$M_s =$		$\frac{1}{2} h (2 - q_s) j_s b t^2 - f'_s p' b d r \sin \alpha \cos \alpha$		(b')
$M_s =$		$-\frac{1}{2} h (2 - q_s) j'_s b t^2 + f_s p b d r \sin \alpha \cos \alpha$		(c')
$j_s =$		$\frac{d}{t} - \frac{3 - 2q}{6 - 3q} = \frac{r}{t} - j'$		(d')
$j'_s =$		$\frac{3 - 2q}{6 - 3q} - \frac{d'}{t} = \frac{r}{t} - j$		(e')
$a_s =$		$-\left(1 - q_s \frac{d}{t}\right)$		(f')
$a'_s =$		$-\left(1 - q_s \frac{d'}{t}\right)$		(g')
$c_s =$		$-\left(\frac{d}{t}\right)\left(\frac{1 - q_s \frac{d}{t}}{2 - q_s}\right)$		(h')
$c'_s =$		$-\left(\frac{d}{t}\right)\frac{1 - q_s \frac{d'}{t}}{2 - q_s}$		(i')
$g_s =$		$\left(\frac{d}{t}\right)\frac{f_s \sin \alpha \cos \alpha}{n (2 - q_s)}$		(j')
$g'_s =$		$\left(\frac{d}{t}\right)\frac{f'_s \sin \alpha \cos \alpha}{n (2 - q_s)}$		(k')

FOR PARTIAL MOMENTS.

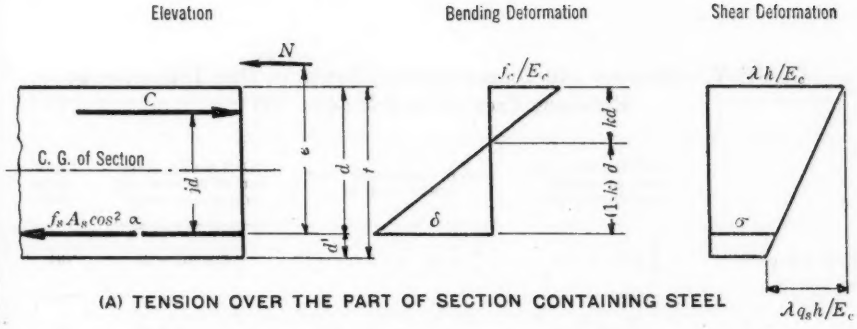
$M_{sc} =$		$\frac{1}{2} h (2 - q_s) j_s t^2$ or $\frac{1}{2} h (2 - q_s) j'_s t^2$		(l')
$M - M_{sc} =$		$f_s p_2 b d r \sin \alpha \cos \alpha$ or $f'_s p'_2 b d r \sin \alpha \cos \alpha$		(m')
$M - M_{sc} =$		$f'_s p' b d r \sin \alpha \cos \alpha$ or $f_s p b d r \sin \alpha \cos \alpha$		(n')
$\frac{f_s}{n} =$		$\sum\sum\sum f_c a \cos^2 \alpha + \lambda h a_s \sin \alpha \cos \alpha$		(p)
$\frac{f'_s}{n} =$		$\sum\sum\sum f_c a' \cos^2 \alpha + \lambda h a'_s \sin \alpha \cos \alpha$		
$g =$		$\sum\sum\sum c \cos^4 \alpha + \lambda c_s \sin^2 \alpha \cos^2 \alpha$		(q)
$g' =$		$\sum\sum\sum c' \cos^4 \alpha + \lambda c'_s \sin^2 \alpha \cos^2 \alpha$		
$\frac{1}{2n} =$		$\sum (p c + p' c') \cos^4 \alpha + \lambda (p c_s + p' c'_s) \sin^2 \alpha \cos^2 \alpha$		(r)

CASE IV.—Bending and Direct Stress; Steel in One Direction in Intrados Face Only (See Fig. 33)

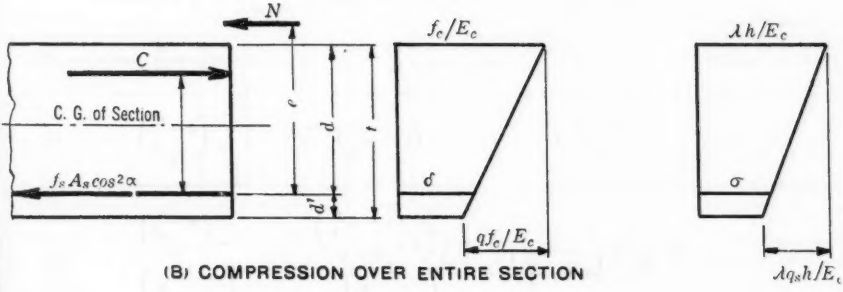
	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$f_s\, p\, b\, d \cos^2 \alpha =$	$\frac{1}{2} f_c\, k\, b\, d - N$	$\frac{1}{2} f_c\, (2-q)\, b\, t - N$	$\frac{1}{2} f_c\, k\, b\, d - N$	(*a*)
$N\, e =$	$\frac{1}{2} f_c\, k\, j\, b\, d^2$	$\frac{1}{2} f_c\, (2-q)\, j\, b\, t^2$	$\frac{1}{2} f_c\, k\, j\, b\, d^2$	(*b*)
	$N\,(e - j\,d) = \frac{f_s\, p\, j\, b\, d^2}{\cos^2 \alpha}$	$N\,(e - j\,t) = \frac{f_s\, p\, j\, b\, d^2}{\cos^2 \alpha}$	$N\,(e - j\,d) = \frac{f_s\, p\, j\, b\, d^2}{\cos^2 \alpha}$	(*c*)
$j =$	$1 - \frac{k}{3}$	$\frac{d}{t} - \frac{3-2q}{6-3q}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a =$	$\frac{1-k}{k}$	$-\left(1 - q\,\frac{d}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*e*)
$c =$	$\left(\frac{1-k}{k^2}\right)\left(\frac{1}{1 - j\,\frac{d}{e}}\right)$	$-\left(\frac{d}{t}\right)\left(\frac{1 - q\,\frac{d}{t}}{2-q}\right)\left(\frac{1}{1 - j\,\frac{t}{e}}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k^2}\right)\left(\frac{1}{1 - j\,\frac{d}{e}}\right)$	(*f*)
$\frac{1}{2\,p\,n} =$	$\frac{f_s \cos^2 \alpha}{n\, f_c\, k \left(1 - j\,\frac{d}{e}\right)}$	$\frac{f_s \cos^2 \alpha}{n\, f_c\, (2-q)\left(1 - j\,\frac{t}{e}\right)\left(\frac{d}{t}\right)}$	$\frac{f_s \cos^2 \alpha}{n\, f_c\, k \left(1 - j\,\frac{d}{e}\right)}$	(*g*)
$f_s\, p\, b\, d =$	$\frac{N\left(\frac{e}{j}\,d - 1\right)}{\cos^2 \alpha}$	$\frac{N\left(\frac{e}{j}\,t - 1\right)}{\cos^2 \alpha}$	$\frac{N\left(\frac{e}{j}\,d - 1\right)}{\cos^2 \alpha}$	(*h*)
	$k = 3\,(1 - j)$	$q = \frac{3 + 6\,j - 6\,\frac{d}{t}}{2 + 3\,j - 3\,\frac{d}{t}}$	$k = 3\left(j + \frac{d'}{d}\right)$	(*i*)

CASE IV.—(*Continued.*)

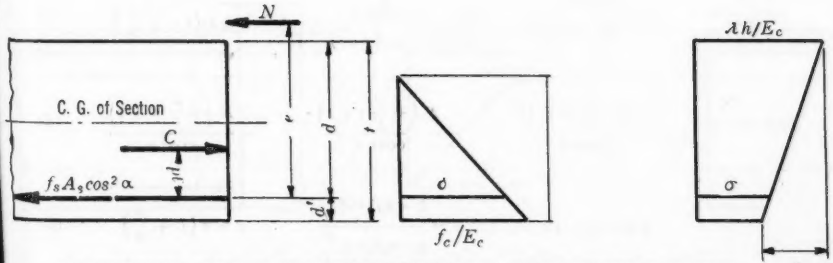
	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$f_s\, p\, b\, d \sin \alpha \cos \alpha =$		$\frac{1}{2}\, h\, (2 - q_s)\, b\, t - S$		(a')
$S\, e_s =$		$\frac{1}{2}\, h\, (2 - q_s)\, j_s\, t^2$		(b')
$S\, (e_s - j\, t) =$		$f_s\, p\, j_s\, b\, d\, t \sin \alpha \cos \alpha$		(c')
$j_s =$		$\frac{d}{t} - \frac{3 - 2\, q_s}{6 - 3\, q_s}$		(d')
$a_s =$		$-\left(1 - q_s \frac{d}{t}\right)$		(e')
$c_s =$		$-\left(\frac{d}{t}\right)\left(\frac{1 - q_s \frac{d}{t}}{2 - q_s}\right)\left(\frac{1}{1 - j_s \frac{t}{e_s}}\right)$		(f')
$\frac{1}{2\, p\, n} =$		$\left(\frac{d}{t}\right) \frac{f_s \sin \alpha \cos \alpha}{n\, (2 - q_s)\left(1 - j_s \frac{t}{e_s}\right)}$		(g')
$f_s\, p\, b\, d =$		$\frac{S\left(\frac{e_s}{j_s}\, t - 1\right)}{\sin \alpha \cos \alpha}$		(h')
$q_s =$		$\frac{3 + 6\, j_s - 6\, \frac{d}{t}}{2 + 3\, j_s - 3\, \frac{d}{t}}$		(i')
$\frac{f_s}{n} =$		$\sum f_c\, a \cos^2 \alpha + \lambda\, h\, a_s \sin \alpha \cos \alpha$		(j)
$\frac{1}{2\, p\, n} =$		$\sum c \cos^4 \alpha + \lambda\, c_s \sin^2 \alpha \cos^2 \alpha$		(k)



(A) TENSION OVER THE PART OF SECTION CONTAINING STEEL



(B) COMPRESSION OVER ENTIRE SECTION

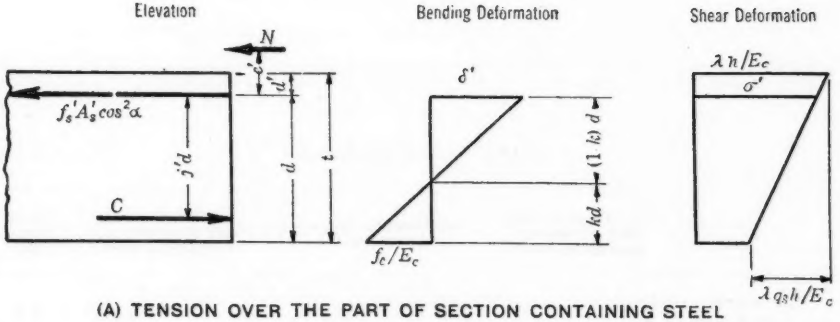


(C) TENSION OVER THE PART OF SECTION NOT CONTAINING STEEL

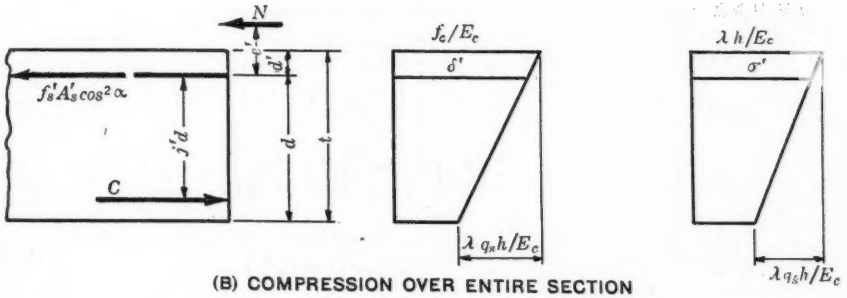
FIG. 33.—BENDING AND DIRECT STRESS; STEEL IN ONE DIRECTION IN INTRADOS FACE ONLY. (SEE CASE IV FOR STEEL IN ONE DIRECTION AND CASE X FOR STEEL IN TWO DIRECTIONS.)

CASE V.—Bending and Direct Stress; Steel in One Direction in Extrados Face Only (See Fig. 34)

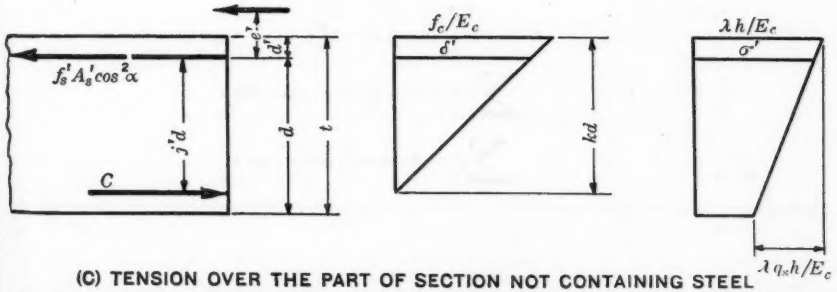
	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$f'_s p' b d \cos^2 \alpha =$	$\frac{1}{2} f_c k b d - N$	$\frac{1}{2} f_c (2-q) b t - N$	$\frac{1}{2} f_c k b d - N$	(*a*)
$N e' =$	$-\frac{1}{2} f_c k j' b d^2$	$-\frac{1}{2} f_c (2-q) j' b t^2$	$-\frac{1}{2} f_c k j' b d^2$	(*b*)
	$N (e' + j' d) = -f'_s p' b d^2 \cos^2 \alpha$	$N (e' + j' t) = -f'_s p' b d t \cos^2 \alpha$	$N (e' + j d) = -f'_s p' j' b d^2 \cos^2 \alpha$	(*c*)
$j' =$	$1 - \frac{k}{3}$	$\frac{3-2q}{6-3q} - \frac{d'}{t}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a' =$	$\frac{1-k}{k}$	$-\left(1 - q\frac{d'}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*e*)
$c' =$	$\left(\frac{1-k_2}{k^2}\right)\left(\frac{1}{1+j'\frac{d}{e'}}\right)$	$-\left(\frac{d}{t}\right)\left(\frac{1-q\frac{d'}{t}}{2-q}\right)\left(\frac{1}{1+j'\frac{t}{e'}}\right)$	$-\left(\frac{k-\frac{d'}{d}}{k^2}\right)\left(\frac{1}{1+j'\frac{d}{e'}}\right)$	(*f*)
$\frac{1}{2p'n} =$	$\frac{f'_s \cos^2 \alpha}{n f_c k \left(1+j\frac{d}{e'}\right)}$	$\frac{\left(\frac{d}{t}\right) f'_s \cos^2 \alpha}{n f_c (2-q)\left(1+j'\frac{t}{e'}\right)}$	$\frac{f'_s \cos^2 \alpha}{n f_c k \left(1+j'\frac{d}{e'}\right)}$	(*g*)
$f'_s p' b d =$	$-\frac{N\left(+\frac{e'}{j'}d+1\right)}{\cos^2 \alpha}$	$-\frac{N\left(+\frac{e'}{j'}t+1\right)}{\cos^2 \alpha}$	$-\frac{N\left(+\frac{e'}{j'}d+1\right)}{\cos^2 \alpha}$	(*h*)
	$k = 3(1-j')$	$q = \frac{3-6j'-6\frac{d'}{t}}{2-3j'-3\frac{d'}{t}}$	$k = 3\left(j' + \frac{d'}{d}\right)$	(*i*)



(A) TENSION OVER THE PART OF SECTION CONTAINING STEEL



(B) COMPRESSION OVER ENTIRE SECTION



(C) TENSION OVER THE PART OF SECTION NOT CONTAINING STEEL

FIG. 34.—BENDING AND DIRECT STRESS; STEEL IN EXTRADOS FACE ONLY. (SEE CASE V FOR STEEL IN ONE DIRECTION AND CASE XI FOR STEEL IN TWO DIRECTIONS.)

CASE V.—*(Continued.)*

	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$\frac{f'_s p' d \sin \alpha}{\cos \alpha} =$		$\frac{1}{2} h (2 - q_s) t - S$		(a')
$S e'_s =$		$-\frac{1}{2} h (2 - q_s) j'_s b t^2$		(b')
$S (e'_s + j'_s t) =$		$-f'_s p' j'_s b d t \sin \alpha \cos \alpha$		(c')
$j'_s =$		$\frac{3 - 2 q_s}{6 - 3 q_s} - \frac{d'}{t}$		(d')
$a'_s =$		$-\left(1 - q_s \frac{d'}{t}\right)$		(e')
$c'_s =$		$-\left(\frac{d}{t}\right)\left(\frac{1 - q_s \frac{d'}{t}}{2 - q_s}\right)\left(\frac{1}{1 + j' \frac{t}{e'_s}}\right)$		(f')
$\frac{1}{2 p' n} =$		$-\left(\frac{d}{t}\right)\frac{f'_s \sin \alpha \cos \alpha}{h (2 - q_s)\left(1 + j'_s \frac{t}{e'_s}\right)}$		(g')
$f'_s p' b d =$		$-\frac{S\left(\frac{e_s}{j_s} t + 1\right)}{\sin \alpha \cos \alpha}$		(h')
$q_s =$		$\frac{3 - 6 j'_s - 6 \frac{d'}{t}}{2 - 3 j'_s - 3 \frac{d'}{t}}$		(i')
$\frac{f'_s}{n} =$		$\sum f_c a' \cos^2 \alpha + \lambda h a'_s \sin \alpha \cos \alpha$		(j)
$\frac{1}{2 p' n} =$		$\sum c' \cos^2 \alpha + \lambda c'_s \sin^2 \alpha \cos^2 \alpha$		(k)

CASE VI.—Bending and Direct Stress; Steel in One Direction Only but in Both Faces (See Fig. 35)

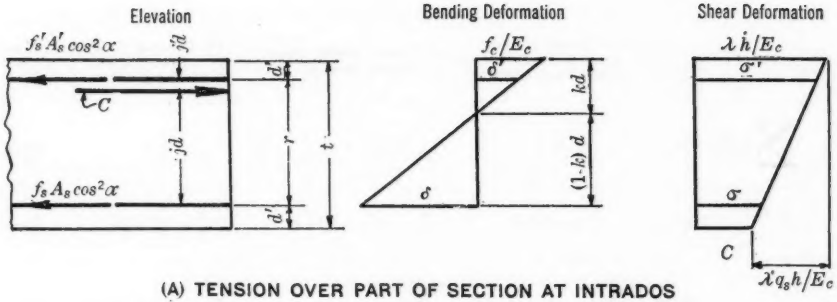
		(*A*) Tension over part of section at intrados	(*B*) Compression over entire section	(*C*) Tension over part of section at extrados	Equations (64)
$(f_s p + f'_s p')\ b\ d \cos^2 \alpha =$		$\frac{1}{2} f_c\ k\ b\ d - N$	$\frac{1}{2} f_c\ (2-q)\ b\ t - N$	$\frac{1}{2} f_c\ k\ b\ d - N$	(*a*)
$N\ e =$		$\frac{1}{2} f_c\ k\ j\ b\ d^2 - f'_s p'\ b\ d\ r \cos^2 \alpha$	$\frac{1}{2} f_c\ (2-q)\ j\ b\ t^2 - f'_s p'\ b\ d\ r \cos^2 \alpha$	$\frac{1}{2} f_c\ k\ j\ b\ d^2 - f'_s p'\ b\ d\ r \cos^2 \alpha$	(*b*)
$N\ e' =$		$-\frac{1}{2} f_c\ k\ j'\ b\ d^2 + f_s p\ b\ d\ r \cos^2 \alpha$	$-\frac{1}{2} f_c\ (2-q)\ j'\ b\ t^2 + f_s p\ b\ d\ r \cos^2 \alpha$	$-\frac{1}{2} f_c\ k\ j'\ b\ d^2 + f_s p\ b\ d\ r \cos^2 \alpha$	(*c*)
$j =$		$1 - \frac{k}{3} = \frac{r}{d} - j'$	$\frac{d}{t} - \frac{3-2q}{6-3q} = \frac{r}{t} - j'$	$\frac{k}{3} - \frac{d'}{d} = \left(\frac{r}{d} - j'\right)$	(*d*)
$j' =$		$\frac{k}{3} - \frac{d'}{d} = \frac{r}{d} - j$	$+\frac{3-2q}{6-3q} - \frac{d'}{t} = +\frac{r}{t} - j$	$1 - \frac{k}{3} = \frac{r}{d} - j$	(*e*)
$a =$		$\frac{1-k}{k}$	$-\left(1 - q\,\frac{d}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*f*)
$a' =$		$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	$-\left(1 - q\,\frac{d'}{t}\right)$	$\frac{1-k}{k}$	(*g*)
$c =$		$\left(\frac{1-k}{k^2}\right)\left(\frac{1}{1 - j\,\frac{d}{e}}\right)$	$-\left(\frac{1 - q\frac{d}{t}}{2-q}\right)\left(\frac{1}{1 - j\,\frac{t}{e}}\right)\frac{d}{t}$	$-\left(\frac{k - \frac{d'}{d}}{k^2}\right)\left(\frac{1}{1 - j\,\frac{d}{e}}\right)$	(*h*)
$c' =$		$-\left(\frac{k - \frac{d'}{d}}{k^2}\right)\left(\frac{1}{1 + j'\,\frac{d}{e'}}\right)$	$-\left(\frac{1 - q\frac{d'}{t}}{2-q}\right)\left(\frac{1}{1 + j'\,\frac{t}{e'}}\right)\frac{d}{t}$	$\left(\frac{1-k}{k^2}\right)\left(\frac{1}{1 + j'\,\frac{d}{e'}}\right)$	(*i*)
$g =$		$\frac{f_s \cos^2 \alpha}{n\ f_c\ k\left(1 - j\,\frac{d}{e}\right)}$	$\frac{d}{t}\,\frac{f_s \cos^2 \alpha}{n f_c (2-q)\left(1 - j\,\frac{t}{e}\right)}$	$\frac{f_s \cos^2 \alpha}{n\ f_c\ k\left(1 - j\,\frac{d}{e}\right)}$	(*j*)
$g' =$		$\frac{f'_s \cos^2 \alpha}{n\ f_c\ k\left(1 + j'\,\frac{d}{e'}\right)}$	$\frac{d}{t}\,\frac{f'_s \cos^2 \alpha}{n f_c (2-q)\left(1 - j'\,\frac{t}{e'}\right)}$	$\frac{f'_s \cos^2 \alpha}{n\ f_c\ k\left(1 + j'\,\frac{d}{e'}\right)}$	(*k*)
For partial moments	$N_c\ e$	$\frac{1}{2} f_c\ k\ j\ b\ d^2$	$\frac{1}{2} f_c\ (2-q)\ j\ b\ t^2$		(*l*)
	— or — $N_c\ e'$		$-\frac{1}{2} f_c\ (2-q)\ j'\ b\ t^2$	$-\frac{1}{2} f_c\ k\ j'\ b\ d^2$	(*m*)
	$(N - N_c)\ e$	$-f'_s p'\ b\ d\ r \cos^2 \alpha$	$-f'_s p'\ b\ d\ r \cos^2 \alpha$ or $-f'_s p'_2\ b\ d\ r \cos^2 \alpha$	$-f'_s\ p'_2\ b\ d\ r \cos^2 \alpha$	(*n*)
	$(N - N_c)\ e'$	$f_s p_2\ b\ d\ r \cos^2 \alpha$	$f_s p_2\ b\ d\ r \cos^2 \alpha$ or $f_s p\ b\ d\ r \cos^2 \alpha$	$f_s p\ b\ d\ r \cos^2 \alpha$	(*p*)

CASE VI.—(*Continued.*)

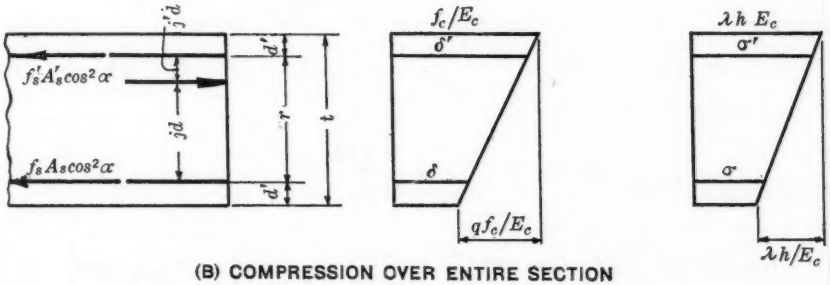
	(*A*) Tension over part of section at intrados	(*B*) Compression over entire section	(*C*) Tension over part of section at extrados	Equations (64)
$(f_s\,p + f'_s\,p')\,b\,d \sin\alpha\cos\alpha =$		$\frac{1}{2}\,h\,(2-q_s)\,b\,t - S$		(a')
$S_s\,e_s =$		$\frac{1}{2}\,h\,(2-q_s)\,j_s\,b\,t^2 - f'_s\,p'\,b\,d\,r\sin\alpha\cos\alpha$		(b')
$S_s\,e'_s =$		$-\frac{1}{2}\,h\,(2-q_s)\,j'_s\,b\,t^2 + f_s\,p\,b\,d\,r\sin\alpha\cos\alpha$		(c')
$j_s =$		$\frac{d}{t} - \frac{3-2\,q_s}{6-3\,q_s}$		(d')
$j'_s =$		$\frac{3-2\,q}{6-3\,q} - \frac{d'}{t} = \frac{r}{t} - j$		(e')
$a_s =$		$-\left(1 - q_s\frac{d}{t}\right)$		(f')
$a'_s =$		$-\left(1 - q_s\frac{d'}{t}\right)$		(g')
$c_s =$		$-\left(\frac{d}{t}\right)\left(\frac{1-q_s\frac{d}{t}}{2-q_s}\right)\left(\frac{1}{1-j_s\frac{t}{e_s}}\right)$		(h')
$c'_s =$		$-\left(\frac{d}{t}\right)\left(\frac{1-q_s\frac{d'}{t}}{2-q_s}\right)\left(\frac{1}{1+j'_s\frac{t}{e'_s}}\right)$		(i')
$g =$		$\left(\frac{d}{t}\right)\frac{f_s\sin\alpha\cos\alpha}{n\,h\,(2-q_s)\left(1-j_s\frac{t}{e_s}\right)}$		(j')
$g' =$		$\left(\frac{d}{t}\right)\frac{f'_s\sin\alpha\cos\alpha}{n\,h\,(2-q_s)\left(1+j_s\frac{t}{e'_s}\right)}$		(k')

FOR PARTIAL MOMENTS

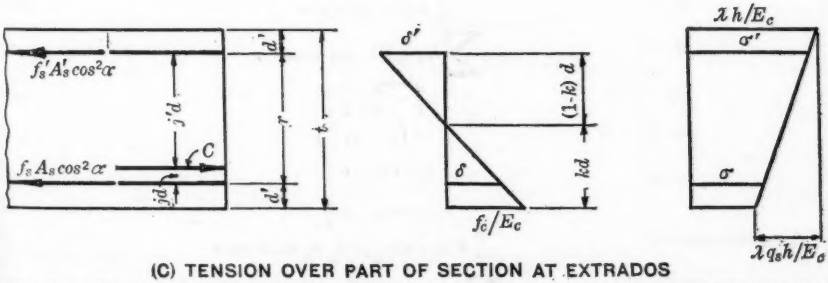
$S_c\,e_s =$		$\frac{1}{2}\,h\,(2-q_s)\,j_s\,b\,t^2$		(l')
$S_c\,e'_s =$		$-\frac{1}{2}\,h\,(2-q_s)\,j'_s\,b\,t^2$		(m')
$(S-S_c)\,e_s =$		$-f'_s\,p'\,b\,d\,r\cos^2\alpha$ or $-f'_s\,p'_2\,b\,d\,r\cos^2\alpha$		(n')
$(S-S_c)\,e'_s =$		$f_s\,p_2\,b\,d\,r\cos^2\alpha$ or $f_s\,p\,b\,d\,r\cos^2\alpha$		(p')
$\frac{f_s}{n} =$		$\sum\sum f_c\,a\cos^2\alpha + \lambda\,h\,a_s\sin\alpha\cos\alpha$		(q)
$\frac{f'_s}{n} =$		$\sum\sum f_c\,a\cos^2\alpha + \lambda\,h\,a'_s\sin\alpha\cos\alpha$		
$g =$		$\sum\sum c\,\cos^4\alpha + \lambda\,c_s\sin^2\alpha\cos^2\alpha$		(m)
$g' =$		$\sum\sum c'\,\cos^4\alpha + \lambda\,c'_s\sin^2\alpha\cos^2\alpha$		
$\frac{1}{2\,n} =$		$\sum (p\,c + p'\,c')\cos^4\alpha + \lambda\,(p\,c_s + p'\,c'_s)\,\sin^2\alpha\cos^2\alpha$		(n)



(A) TENSION OVER PART OF SECTION AT INTRADOS



(B) COMPRESSION OVER ENTIRE SECTION



(C) TENSION OVER PART OF SECTION AT EXTRADOS

FIG. 35.—BENDING AND DIRECT STRESS; STEEL IN BOTH FACES. (SEE CASE VI FOR STEEL IN ONE DIRECTION AND CASE XII FOR STEEL IN TWO DIRECTIONS.)

CASE VII.—Simple Bending; Steel in Two Directions in Intrados Face Only (See Fig. 30)

	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$\sum f_s p \cos^2 \alpha =$	$\frac{1}{2} f_c k$	$\frac{1}{2} f_c (2-q) \frac{t}{d}$	$\frac{1}{2} f_c k$	(*a*)
$M =$	$\frac{1}{2} f_c k j b d^2$	$\frac{1}{2} f_c (2-q) j b t^2$	$\frac{1}{2} f_c k j b d^2$	(*b*)
$M =$	$\sum f_s p j b d^2 \cos^2 \alpha$	$\sum f_s p j b d t \cos^2 \alpha$	$\sum f_s p j b d^2 \cos^2 \alpha$	(*c*)
$j =$	$1 - \frac{k}{3}$	$\frac{d}{t} - \frac{3-2q}{6-3q}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a =$	$\frac{1-k}{k}$	$-\left(1 - q \frac{d}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*e*)
	$k = 3(1-j)$	$q = \frac{3 + 6j - 6\frac{d}{t}}{2 + 3j - 3\frac{d}{t}}$	$k = 3\left(j + \frac{d'}{d}\right)$	(*f*)
$f_\alpha p_\alpha \cos^2 \alpha =$	$\frac{1}{2} f_c k - f_\beta p_\beta \cos^2 \beta$	$\frac{1}{2} f_c (2-q) \frac{t}{d} - f_\beta p_\beta \cos^2 \beta$	$\frac{1}{2} f_c k - f_\beta p_\beta \cos^2 \beta$	(*g*)
$\sum f_s p \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) \frac{t}{d}$			(*a'*)
$M_s =$	$\frac{1}{2} h (2 - q_s) b j_s t^2$			(*b'*)
$M_s =$	$\sum f_s p j_s b d t \sin \alpha \cos \alpha$			(*c'*)
$j_s =$	$\frac{d}{t} - \frac{3-2q_s}{6-3q_s}$			(*d'*)
$a_s =$	$-\left(1 - q_s \frac{d}{t}\right)$			(*e'*)
$q_s =$	$\frac{3 + 6 j_s - 6 \frac{d}{t}}{3 + 3 j_s - 3 \frac{d}{t}}$			(*f'*)
$f_\alpha p_\alpha \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) \frac{t}{d} - f_\beta p_\beta \sin \beta \cos \beta$			(*g'*)
$\frac{f_s}{n} =$	$\sum f_c a \cos^2 \alpha + \lambda h a_s \sin \alpha \cos \alpha$			(*h*)

CASE VIII.—Simple Bending; Steel in Two Directions in Extrados Face Only (See Fig. 31)

	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64).
$\sum f'_s p' \cos^2 \alpha =$	$\frac{1}{2} f_c k$	$\frac{1}{2} f_c (2-q) \frac{t}{d}$	$\frac{1}{2} f_c k$	(*a*)
$M =$	$-\frac{1}{2} f_c k j' b d^2$	$-\frac{1}{2} f_c (2-q) j' b t^2$	$-\frac{1}{2} f_c k j' b d^2$	(*b*)
$M =$	$-\sum f'_s p' j' b d^2 \cos^2 \alpha$	$-\sum f'_s p' j' b d t \cos^2 \alpha$	$-\sum f'_s p' j' b d^2 \cos^2 \alpha$	(*c*)
$j' =$	$\left(1 - \frac{k}{3}\right)$	$\frac{3-2q}{6-3q} - \frac{d'}{t}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a' =$	$\frac{1-k}{k}$	$-\left(1 - q\frac{d'}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*e*)
	$k = 3(1-j')$	$q = \frac{3 - 6j' - 6\frac{d'}{t}}{2 - 3j' - 3\frac{d'}{t}}$	$k = 3\left(j' + \frac{d'}{d}\right)$	(*f*)
$f_\alpha p_\alpha \cos^2 \alpha =$	$\frac{1}{2} f_c k - f_\beta p_\beta \cos^2 \beta$	$\frac{1}{2} f_c (2-q) \frac{t}{d} - f_\beta p_\beta \cos^2 \beta$	$\frac{1}{2} f_c k - f_\beta p_\beta \cos^2 \beta$	(*g*)
$\sum f'_s p' \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) \frac{t}{d}$			(*a'*)
$M_s =$	$-\frac{1}{2} h (2 - q_s) b j'_s t^2$			(*b'*)
$M_s =$	$-\sum f'_s p' j'_s b d t^2 \sin \alpha \cos \alpha$			(*c'*)
$j_s =$	$\frac{3-2q_s}{6-3q} - \frac{d'}{t}$			(*d'*)
$a'_s =$	$-\left(1 - q_s \frac{d'}{t}\right)$			(*e'*)
$q_s =$	$\frac{3 - 6j' - 6\frac{d'}{t}}{2 - 3j' - 3\frac{d'}{t}}$			(*f'*)
$f_\alpha p_\alpha \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) \frac{t}{d} - f_\beta p_\beta \sin \beta \cos \beta$			(*g'*)
$\frac{f_s}{n} =$	$\sum f_c a' \cos^2 \alpha + \lambda h a_s \sin \alpha \cos \alpha$			(*h*)

CASE IX.—Simple Bending; Steel in Two Directions and in Both Faces (See Fig. 32)

	(*A*) Tension over part of section at intrados	(*B*) Compression over entire section	(*C*) Tension over the part of section at extrados	Equations (64)
$\sum \frac{(f_s p + f'_s p')}{\cos^2 \alpha} =$	$\frac{1}{2} f_c k$	$\frac{1}{2} f_c (2-q) \frac{t}{d}$	$\frac{1}{2} f_c k$	(*a*)
$M =$	$\frac{1}{2} f_c k j b d^2 - \sum f'_s p' b d r \cos^2 \alpha$	$\frac{1}{2} f_c (2-q) j b t^2 - \sum f'_s p' b d r \cos^2 \alpha$	$\frac{1}{2} f_c k j b d^2 - \sum f'_s p' b d r \cos^2 \alpha$	(*b*)
$M =$	$-\frac{1}{2} f_c k j' b d^2 + \sum f_s p b d r \cos^2 \alpha$	$-\frac{1}{2} f_c (2-q) j' b t^2 + \sum f_s p b d r \cos^2 \alpha$	$-\frac{1}{2} f_c k j' b d^2 + \sum f_s p b d r \cos^2 \alpha$	(*c*)
$j =$	$1 - \frac{k}{3} = \frac{r}{d} - j'$	$\frac{d}{t} - \frac{3-2q}{6-3q} = \frac{r}{t} - j'$	$\frac{k}{3} - \frac{d'}{d} = \left(\frac{r}{d} - j'\right)$	(*d*)
$j' =$	$\frac{k}{3} - \frac{d'}{d} = \frac{r}{d} - j$	$+\frac{3-2q}{6-3q} - \frac{d'}{t} = +\frac{r}{t} - j$	$1 - \frac{k}{3} = \frac{r}{d} - j$	(*e*)
$a =$	$\frac{1-k}{k}$	$-\left(1 - q\frac{d}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*f*)
$a' =$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	$-\left(1 - q\frac{d'}{t}\right)$	$\frac{1-k}{k}$	(*g*)
$\frac{(f_\alpha p_\alpha + f'_\alpha p'_\alpha)}{\cos^2 \alpha} =$	$\frac{1}{2} f_c k - (f_\beta p_\beta + f'_\beta p'_\beta) \cos^2 \beta$	$\frac{1}{2} f_c (2-q) \frac{t}{d} - \frac{(f_\beta p_\beta + f'_\beta p'_\beta)}{\cos^2 \beta}$	$\frac{1}{2} f_c k - (f_\beta p_\beta + f'_\beta p'_\beta) \cos^2 \beta$	(*h*)

CASE IX.—(*Continued.*)

	(*A*) Tension over part of section at intrados	(*B*) Compression over entire section	(*O*) Tension over the part of section of extrados	Equations (64)
$\sum \frac{(f_s p + f'_s p')}{\sin \alpha \cos \alpha} =$		$\frac{1}{2} h (2 - q_s) \frac{t}{d}$		(*a'*)
$M_s =$		$\frac{1}{2} h (2 - q_s) j_s b t^2 - \sum f'_s p' b d r \sin \alpha \cos \alpha$		(*b'*)
$M_s =$		$-\frac{1}{2} h (2 - q_s) j'_s b t^2 + \sum f_s p b d r \sin \alpha \cos \alpha$		(*c'*)
$j_s =$		$\frac{d}{t} - \frac{3 - 2q}{6 - 3q}$		(*d'*)
$j'_s =$		$\frac{3 - 2q}{6 - 3q} - \frac{d'}{t}$		(*e'*)
$a_s =$		$-\left(1 - q_s \frac{d}{t}\right)$		(*f'*)
$a'_s =$		$-\left(1 - q_s \frac{d}{t}\right)$		(*g'*)
$\frac{(f_\alpha p_\alpha + f'_\alpha p'_\alpha)}{\sin \alpha \cos \alpha} =$		$\frac{1}{2} h (2 - q_s) \frac{t}{d} - (f_\beta p_\beta + f'_\beta p'_\beta) \sin \beta \cos \beta$		(*h'*)
$\frac{f_s}{n} =$		$\sum f_c a \cos^2 \alpha + \lambda h a_s \sin \alpha \cos \alpha$		(*i*)
$\frac{f'_s}{n} =$		$\sum f_c a' \cos^2 \alpha + \lambda h a'_s \sin \alpha \cos \alpha$		

CASE X.—Bending and Direct Stress; Steel in Two Directions in Intrados Face Only (See Fig. 33)

	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$\sum f_s p\, b\, d \cos^2 \alpha =$	$\frac{1}{2} f_c\, k\, b\, d - N$	$\frac{1}{2} f_c\, (2-q)\, b\, t - N$	$\frac{1}{2} f_c\, k\, b\, d - N$	(*a*)
$N\, e =$	$\frac{1}{2} f_c\, k\, j\, b\, d^2$	$\frac{1}{2} f_c\, (2-q)\, j\, b\, t^2$	$\frac{1}{2} f_c\, k\, j\, b\, d^2$	(*b*)
	$N\,(e - j\,d) = \sum f_s\, p\, j\, b\, d^2 \cos^2 \alpha$	$N\,(e - j\,t) = \sum f_s\, p\, j\, b\, d^2 \cos^2 \alpha$	$N\,(e - j\,d) = \sum f_s\, p\, j\, b\, d^2 \cos^2 \alpha$	(*c*)
$j =$	$1 - \frac{k}{3}$	$\frac{d}{t} - \frac{3-2q}{6-3q}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a =$	$\frac{1-k}{k}$	$-\left(1 - q\,\frac{d}{t}\right)$	$-\frac{\left(k - \frac{d'}{d}\right)}{k}$	(*e*)
	$k = 3\,(1-j)$	$q = \frac{3 + 6j - 6\frac{d}{t}}{2 + 3j' - 3\frac{d}{t}}$	$k = 3\left(j' + \frac{d'}{d}\right)$	(*f*)
$f_a\ p_a\ b\, d \cos^2 \alpha =$	$\frac{1}{2} f_c\, k\, b\, d - N - f_\beta\ p_\beta\, b\, d \cos^2 \beta$	$\frac{1}{2} f_c\, (2-q)\, b\ t - N - f_\beta\ p_\beta\, b\, d \cos^2 \beta$	$\frac{1}{2} f_c\, k\, b\, d - N - f_\beta\ p_\beta\, b\, d \cos^2 \beta$	(*g*)
$\sum f_s p b d \sin \alpha \cos \alpha =$	$\frac{1}{2} h\, (2 - q_s)\, b\, t - S$			(*a'*)
$S\, e_s =$	$\frac{1}{2} h\, (2 - q_s)\, j_s\, b\, t^2$			(*b'*)
$S\,(e_s - j\,t) =$	$\sum f_s\, p\, j_s\, b\, d\, t \sin \alpha \cos \alpha$			(*c'*)
$j_s =$	$\frac{d}{t} - \frac{3-2q_s}{6-3q_s}$			(*d'*)
$a_s =$	$-\left(1 - q_s\,\frac{d}{t}\right)$			(*e'*)
$q_s =$	$\frac{3 + 6\, j_s - 6\,\frac{d}{t}}{2 + 3\, j_s - 3\,\frac{d}{t}}$			(*f'*)
$f_a\ p_a\ b\, d \sin \alpha \cos \alpha =$	$\frac{1}{2} h\, (2 - q_s)\, b\, t - S - f_\beta\ p_\beta\, b\, d \sin \beta \cos \beta$			(*g'*)
$\frac{f_s}{n} =$	$\sum f_c\, a \cos^2 \alpha + \lambda\, h\, a_s \sin \alpha \cos \alpha$			(*h*)

CASE XI.—Bending and Direct Stress; Steel in Two Directions in Extrados Face Only (See Fig. 34)

	(*A*) Tension over part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over part of section not containing steel	Equations (64)
$\sum f'_s p' b d \cos^2 \alpha =$	$\frac{1}{2} f_c k b d - N$	$\frac{1}{2} f_c (2-q) b t - N$	$\frac{1}{2} f_c k b d - N$	(*a*)
$N e' =$	$-\frac{1}{2} f_c k j' b d^2$	$-\frac{1}{2} f_c (2-q) j' b t^2$	$-\frac{1}{2} f_c k j' b d^2$	(*b*)
$N (e' - j' d) =$	$-\sum f_s p j' b d^2 \cos^2 \alpha$	$-\sum f_s p j' b d t \cos^2 \alpha$	$\sum f_s p j' d^2 \cos^2 \alpha$	(*c*)
$j' =$	$1 - \frac{k}{3}$	$\frac{3-2q}{6-3q} - \frac{d'}{t}$	$\frac{k}{3} - \frac{d'}{d}$	(*d*)
$a' =$	$\frac{1-k}{k}$	$-\left(1 - q\frac{d'}{t}\right)$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	(*e*)
	$k = 3(1-j')$	$q = \frac{3 - 6j' - 6\frac{d'}{t}}{2 - 3j' - 3\frac{d'}{t}}$	$k = 3\left(j' + \frac{d'}{d}\right)$	(*f*)
$f'_\alpha p'_\alpha b d \cos^2 \alpha =$	$\frac{1}{2} f_c k b d - N - f_\beta p_\beta b d \cos^2 \beta$	$\frac{1}{2} f_c (2-q) b t - N - f_\beta p_\beta b d \cos^2 \beta$	$\frac{1}{2} f_c k b d - N - f_\beta p_\beta b d \cos^2 \beta$	(*g*)

$\sum f'_s p' b d \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) b t - S$	(*a'*)
$S e'_s =$	$-\frac{1}{2} h (2 - q_s) j'_s b t^2$	(*b'*)
$S (e'_s + j'_s t) =$	$-\sum f'_s p' j'_s b d t \sin \alpha \cos \alpha$	(*c'*)
$j'_s =$	$\frac{3-2q_s}{6-3q_s} - \frac{d'}{t}$	(*d'*)
$a'_s =$	$-\left(1 - q\frac{d'}{t}\right)$	(*e'*)
$q_s =$	$\frac{3 - 6j'_s - 6\frac{d'}{t}}{2 - 3j'_s - 3\frac{d'}{t}}$	(*f'*)
$f'_\alpha p'_\alpha b d \sin \alpha \cos \alpha =$	$\frac{1}{2} h (2 - q_s) b t - S - f'_\beta p'_\beta b d \sin \beta \cos \beta$	(*g'*)
$\frac{f'_s}{n} =$	$\sum f_c a' \cos^2 \alpha + \lambda h a'_s \sin \alpha \cos \alpha$	(*i*)

CASE XII.—BENDING AND DIRECT STRESS; STEEL IN TWO DIRECTIONS AND IN BOTH FACES (SEE FIG. 35)

	(*A*) Tension over the part of section containing steel	(*B*) Compression over entire section	(*C*) Tension over the part of section not containing steel	Equations (64)
$\sum \frac{(f_s p + f'_s p')}{b\,d\cos^2\alpha} =$	$\frac{1}{2} f_c\, k\, b\, d - N$	$\frac{1}{2} f_c\,(2-q)\, b\, t - N$	$\frac{1}{2} f_c\, k\, b\, d - N$	(*a*)
$N\,e =$	$\frac{1}{2} f_c\, k\, j\, b\, d^2 - \sum f'_s p'\, b\, d\, r \cos^2\alpha$	$\frac{1}{2} f_c\,(2-q)\, j\, b\, t^2 - \sum f'_s p'\, b\, d\, r \cos^2\alpha$	$\frac{1}{2} f_c\, k\, j\, b\, d^2 - \sum f'_s p'\, b\, d\, r \cos^2\alpha$	(*b*)
$N\,e' =$	$-\frac{1}{2} f_c\, k\, j'\, b\, d^2 + \sum f_s p\, b\, d\, r \cos^2\alpha$	$-\frac{1}{2} f_c\,(2-q)\, j'\, b\, t^2 + \sum f_s p\, b\, d\, r \cos^2\alpha$	$-\frac{1}{2} f_c\, k\, j'\, b\, d^2 + \sum f_s p\, b\, d\, r \cos^2\alpha$	(*c*)
$j =$	$1 - \frac{k}{3} = \frac{r}{d} - j'$	$\frac{d}{t} - \frac{3-2q}{6-3q} = \frac{r}{t} - j'$	$\frac{k}{3} - \frac{d'}{d} = \left(\frac{r}{d} - j'\right)$	(*d*)
$j' =$	$\frac{k}{3} - \frac{d'}{d} = \frac{r}{d} - j$	$+\frac{3-2q}{6-3q} - \frac{d'}{t} = +\frac{r}{t} - j$	$1 = \frac{k}{3} = \frac{r}{d} - j$	(*e*)
$a =$	$\frac{1-k}{k}$	$-\left(1 - q\,\frac{d}{t}\right)$	$-\left(k - \frac{d'}{k}\right)$	(*f*)
$a' =$	$-\left(\frac{k - \frac{d'}{d}}{k}\right)$	$-\left(1 - q\,\frac{d'}{t}\right)$	$\frac{1-k}{k}$	(*g*)
$\frac{(f_\alpha\, p_\alpha + f'_\alpha\, p'_\alpha)}{b\,d\cos^2\alpha} =$	$\frac{\frac{1}{2} f_c\, k\, b\, d - N - (f_\beta\, p_\beta + f'_\beta\, p'_\beta)}{b\,d\cos^2\beta}$	$\frac{\frac{1}{2} f_c\,(2-q)\, b\, t - N - (f_\beta\, p_\beta + f'_\beta\, p'_\beta)}{b\,d\cos^2\beta}$	$\frac{\frac{1}{2} f_c\, k\, b\, d - N - (f_\beta\, p_\beta + f'_\beta\, p'_\beta)}{b\,d\cos^2\beta}$	(*h*)
$\sum \frac{(f_s p + f'_s p')}{b\,d\cos^2\alpha} =$	$\frac{1}{2} h\,(2-q_s)\, b\, t - S$			(*a'*)
$S_s e_s =$	$\frac{1}{2} h\,(2-q_s)\, j_s\, b\, t^2 - \sum f'_s p'\, b\, d\, r \cos^2\alpha$			(*b'*)
$S_s e'_s =$	$-\frac{1}{2} h\,(2-q_s)\, j'_s\, b\, t^2 + \sum f_s p\, b\, d\, r \cos^2\alpha$			(*c'*)
$j_s =$	$\frac{d}{t} - \frac{3-2q_s}{6-3q_s}$			(*d'*)
$j'_s =$	$\frac{3-2q}{6-3q} - \frac{d'}{t} = \frac{r}{t} - j_s$			(*e'*)
$a_s =$	$-\left(1 - q_s\,\frac{d}{t}\right)$			(*f'*)
$a'_s =$	$-\left(1 - q_s\,\frac{d'}{t}\right)$			(*g'*)
$\frac{(f_\alpha\, p_\alpha + f'_\alpha\, p'_\alpha)}{b\,d\sin\alpha\cos\alpha} =$	$\frac{1}{2} h\,(2-q_s)\, b\, t - S - (f_\beta\, p_\beta + f'_\beta\, p'_\beta)\sin\beta\cos\beta$			(*h'*)
$\frac{f_s}{n} =$	$\sum f_c\, a \cos^2\alpha + \lambda\, h\, a_s \sin\alpha\cos\alpha$			(*i*)
$\frac{f'_s}{n} =$	$\sum f_c\, a' \cos^2\alpha + \lambda\, h\, a'_s \sin\alpha\cos\alpha$			

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PAPERS

A DISCHARGE DIAGRAM FOR UNIFORM FLOW IN OPEN CHANNELS

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Synopsis

This paper introduces a diagram of discharge stage and slope or fall relation for uniform flow in open channels. It is based on the Chezy formula wherein, for a constant stage, the discharge varies as the square root of the slope or fall and is designed to give the discharges at any stage for the falls that create the resulting discharges.

A discharge diagram consists of a series of fall curves referred to discharges as abscissas and stages as ordinates, from which the discharge for any observed stage and fall can be obtained.

The development of the diagram is shown by a practical application to a series of discharge measurements made in 1914-16 on the Chicago Sanitary District Canal with regulated flows in a uniform rock section.

Discharge diagrams are applicable to all uniform stream flows, whereas the ordinary discharge curve cannot be used where there is back-water effect or where the up-stream and down-stream waters do not rise or fall simultaneously.

The Discharge or Rating Curve

In many streams for uniform flow there is only one stage and a corresponding slope for any discharge in a reach between stations so spaced that the water-surface profile is a straight line. An increase in the discharge creates a higher stage with its corresponding slope. Increments of discharge have their accompanying increments of stage in such a manner that by plotting discharges as abscissas and corresponding stages as ordinates, the points will fall in a regular curve. This is the ordinary discharge curve shown in Fig. 1. The profiles that obtain at the various stages are of the type shown in Fig. 2. They do not cross one another, are nearly parallel, and are approximately parallel to the bed of the stream.

Sometimes, it is necessary to make measurements of discharge when the flow is not uniform; that is, while the stage is rising or falling. Discharges

Note.—Written discussion on this paper will be closed in May, 1931, *Proceedings*.

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thus obtained will fall, respectively, to the right or left of the curve for uniform flow, and methods have been devised for reducing these discharges to their positions on the discharge curve, in the directions indicated in Fig. 3.

When the Discharge Curve Is not Applicable

There are streams in which the flow is varied by inflow and outflow such that, at any stage, a wide range of discharges may obtain. This may be due to various conditions, such as:

(*a*) The back-water effect of high, low, or intermediate stages of the receiving reservoir (river, lake, or pool).
(*b*) The effect of a large tributary inflow.
(*c*) The effect of high, low, or intermediate stages of an up-stream reservoir (lake or pool).

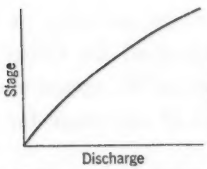

FIG. 1.—TYPICAL DISCHARGE CURVE, UNIFORM FLOW.

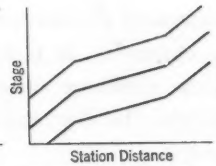

FIG. 2.—PROFILES, TYPE *A*.

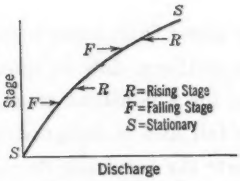

FIG. 3.—TYPICAL DISCHARGE CURVE FROM NON-UNIFORM FLOWS.

A few instances of stream gaugings made under these conditions are given in Table 1.

TABLE 1.—TYPICAL STREAMS THAT HAVE PROFILES OF THE TYPE *B*

Stream	Years	Authorities	Upper reservoir	Lower reservoir	Reaches	Gauging stations
Detroit River*	1902-03.	U. S. Lake Survey.	Lake St. Clair.	Lake Erie.	Fort Wayne, Mich., to Lake St. Clair.	Fort Wayne, Mich.
Illinois River	Various.	U.S. Geological Survey. U.S. Engineering Department. Sanitary District of Chicago.	Lake Peoria.	Mississippi River.	Peoria, Ill., to Pekin, Ill. Copperas Creek to Havana Ill., etc.	Peoria, Ill. Havana, Ill. Beardstown, Ill. Valley City, Ill. Hardin, Ill.
Sanitary District Canal†	1914-16.	Sanitary District of Chicago	Lake Michigan.	Pool above dam at Lockport, Ill.	Willow Springs, Ill., to Lemont, Ill.	Lemont, Ill.

* Report of Chf. of Engrs., U. S. A., for 1902, Appendix EEE, pp. 2867–2872; and for 1903 Appendix EEE, pp. 2813–2817.

† "Hydraulics of Chicago Sanitary District's Main Channel," *Journal*, Western Soc. of Engrs., September, 1920, and discussion, August, 1921.

The instances listed in Table 1 are illustrated graphically in the profiles of Type *B*, Fig. 4. In the reach, *A-B*, the upper profiles have a common mean

elevation of water surface and the lower profiles have another common mean stage.

In making a study and investigation of flood flows of the Illinois River in 1929, these conditions were encountered. It was found that at all the gauging stations in the lower 100 miles of the river, discharges that had been measured by competent observers under the direction of most reliable authorities, would appear to be grossly in error when compared with the values indicated by established discharge curves.

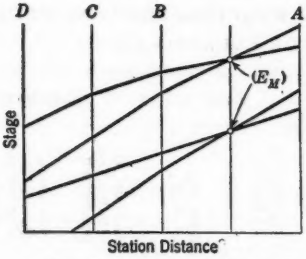

FIG. 4.—PROFILES, TYPE B.

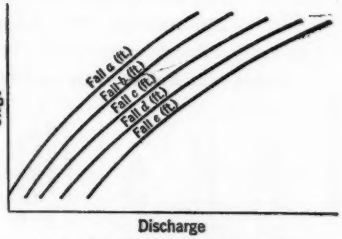

FIG. 5.—TYPICAL DISCHARGE DIAGRAM.

To obtain a substitute for the rating curve under these conditions the writer developed the discharge diagram described herein. It consists of a series of curves of slope which are referred to discharge and stage and from which the discharge can be obtained for a known stage and slope, or fall (see Fig. 5).

DEVELOPMENT OF THE DISCHARGE DIAGRAM

Consider, first, a reach in which the slope is unbroken, and a certain mean stage of reach. The Chezy formula for discharge, Q, in terms of mean area of section, A, mean hydraulic radius, R, and slope, S, is:

$$Q = A\ C\ R^{\frac{1}{2}}\ S^{\frac{1}{2}} = \frac{A\ C\ R^{\frac{1}{2}}\ F^{\frac{1}{2}}}{L^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which, C is a constant and the slope, S, is equal to the fall, F, divided by the length of reach, L. For a single stage under consideration the factors, (A), (C), (R), and (L), are constant. Therefore, K may be substituted in Equation (1) for the quantity, $\frac{A\ C\ R^{\frac{1}{2}}}{L^{\frac{1}{2}}}$, so that Equation (1) becomes,

$$Q = K\ F^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

For any other discharge and corresponding fall (denoted by the subscripts, x),

$$Q_x = K\ F_x^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From Equations (2) and (3), $\frac{Q_x}{Q} = \frac{F_x^{\frac{1}{2}}}{F^{\frac{1}{2}}}$,

or,

$$Q_x = Q\ \frac{\left(F_x^{\frac{1}{2}}\right)}{\left(F^{\frac{1}{2}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and,

$$F_x = F\left(\frac{Q_x}{Q}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The method of obtaining a discharge diagram from these formulas may be described in ten steps, as follows:

1.—Decide upon a base discharge (preferably about an average) to which all observations are to be reduced.

TABLE 2.—SAMPLE COMPUTATIONS REQUIRED FOR CONSTRUCTING DISCHARGE DIAGRAM FOR EQUATION (5)

Group No.	Observation No.	Stage (all minus)			Discharge	Fall		Discharge		
		Observation, E_x, in feet	Observation, E_1, in feet	Mean, E_m, in feet	Observation, and mean, Q, in cubic feet per second	Observation, F, in feet	Base, F_x, in feet	Diagram, Q_x, in cubic feet per second	Residual, $Q_x - Q$, in cubic feet per second	Residual, $\frac{Q_x - Q}{Q}$, percentage
	157	0.68	1.38	1.03	9 300	0.70	0.52	.....	.....	.....
A				1.03	9 300	0.70	0.52	9 400	+100	1.1
	159	0.87	1.55	1.21	8 470	0.68	0.61	.....	.....	.....
B				1.21	8 470	0.68	0.61	8 900	+430	4.8
	164	0.99	1.53	1.26	7 800	0.54	0.57	.....	.....	.....
	156	0.97	1.58	1.27	8 232	0.61	0.58	.....	.....	.....
	158	0.93	1.60	1.27	8 360	0.67	0.52	.....	.....	.....
	155	0.92	1.61	1.25	8 550	0.69	0.60	.....	.....	.....
	165	1.03	1.59	1.31	7 430	0.56	0.65	.....	.....	.....
	60	1.11	1.54	1.33	7 130	0.43	0.54	.....	.....	.....
C				1.23	7 920	0.58	0.58	8 200	+280	2.8
	147	1.16	1.65	1.40	7 370	0.48	0.57	.....	.....	.....
	65	1.21	1.60	1.41	6 750	0.39	0.55	.....	.....	.....
	49	1.23	1.63	1.43	6 900	0.40	0.54	.....	.....	.....
D				1.41	7 010	0.42	0.55	6 600	−410	6.2
	153	1.24	1.81	1.52	7 740	0.57	0.61	.....	.....	.....
	152	1.29	1.77	1.53	7 580	0.48	0.53	.....	.....	.....
	160	1.26	1.86	1.56	8 250	0.60	0.57	.....	.....	.....
E				1.54	7 860	0.55	0.57	7 700	−160	2.1

2.—From each discharge and corresponding fall, using Equation (5), compute the fall corresponding to the base discharge. This step is demonstrated briefly in Table 2 for a number of observations made on the Chicago Drainage Canal in 1914, 1915, and 1916.

3.—Compute the average elevation of the reach for each observation.

4.—Plot the falls as abscissas with the corresponding average stages as ordinates.

5.—Pass an average smooth curve through the points thus obtained and call it the base curve.

6.—From the base curve, take off the fall for the base discharge at desired stage intervals.

7.—By means of Equation (4) compute the discharge corresponding to a desired slope at each of the stage intervals.

8. Compute the discharges corresponding to other desired falls, F_x, at each of the stage intervals.

9.—At the stage intervals plot the discharges as abscissas and the corresponding falls as ordinates.

10.—Join the points of the same slope value, and smooth slope curves will be obtained.

Fig. 6 in an example of a discharge diagram constructed from actual observations made on the Chicago Drainage Canal, some of which data are given in Table 2. The series of slope curves form the discharge diagram (see Fig. 7).

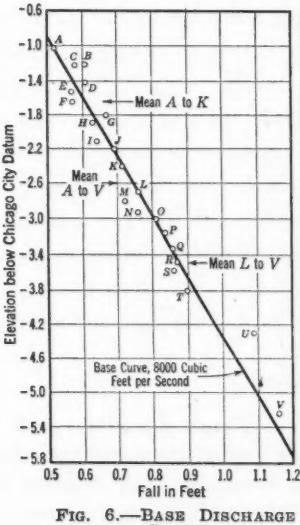

FIG. 6.—BASE DISCHARGE CURVE.

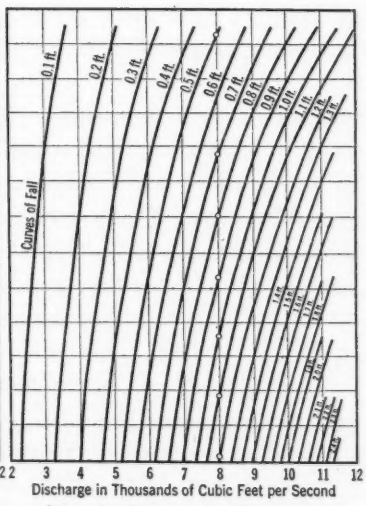

FIG. 7.—DISCHARGE DIAGRAM.

Since a stage at one end of a reach in connection with any certain slope or fall corresponds to a definite discharge, a diagram of discharge, and end-station stage, and a slope or fall relation can be constructed in a manner similar to that for the diagram of discharge, mean stage, and slope, or fall relation, herein described.

CONCLUSIONS

The construction of a discharge diagram involves the additional expense, over the discharge curve, of maintaining and reading two gauges in each reach

instead of one. It would seem that in most cases the more reliable results obtained from the former would warrant the expenditure.

Even if the diagram is constructed for all cases of uniform flow there would also be the special uses for the discharge curve, which gives for any stage the discharge that normally or most frequently occurs. It is often desirable to know the increment of normal discharge for a rise or fall in stage. Thus, the increment of discharge in the outlet of a large lake, taking into account also the effect of evaporation, etc., is a measure of the effect of withdrawal from the lake of a volume equivalent to the increment.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

LEGAL PROBLEMS INVOLVED IN ESTABLISHING SET-BACK LINES[1]

By Clifton Williams,[2] Esq.

Synopsis

When fire, flood, disease, or other calamities threaten the people of a community, National, State, and personal property may be sacrificed without compensation. This paper calls attention to a new idea, based on the age-old function of sovereignty; namely, that if narrow streets in a community become so dangerous as to require widening for the preservation of public health and safety, the acquirement of property for such widening is properly a function of sovereignty.

If the Courts could be convinced by engineers that street-widening problems are indeed problems of sovereignty, the tremendous financial problems encountered at present would be solved. Set-back lines could be established on narrow streets by giving due notice that in ten, twenty, or thirty years from that date the property thus segregated would revert to public ownership. In the meantime, the private owner would have time to make the necessary adjustments to meet the requirements of such a law.

Introduction

The term, "set-back lines", as used in this paper, refers to those established for the purpose of future widening of streets for traffic and not to the so-called set-back lines employed in zoning which were established to mark bounds beyond which buildings shall not be constructed.

Later, the so-called zoning lines or building lines are mentioned in connection with some Supreme Court decisions which established the validity of such lines, without compensation to the abutting owner. The set-back line for street widening is only one step farther and is as clearly within the field of governmental sovereignty as the other form.

The main purpose of this paper is to help engineers realize that the Courts are not trifling with this subject and that lawyers are not trifling with the

Note.—Written discussion on this paper will be closed in May, 1931, *Proceedings*.

[1] Presented at the meeting of the City Planning Division, Milwaukee, Wis., July 11, 1929.

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Courts when they maintain that the street-widening problems of America are just as much problems of sovereignty as fire .ghting, quarantine to prevent the spread of disease, the abatement of nuisances, the struggle to prevent the spread of crime, etc.

Police Power and Set-Back Lines

This effort is prompted by repeated remarks made to the writer by city planning engineers and others who are devoting the best years of their lives to this work, to the effect that they are "in a fog" in attempting to understand the legal somersaults that are apparently going on throughout the United States. Even law students do not understand the term, "police power", and they do not understand clearly how there can be a phase of sovereignty involved in this problem, which is older than any constitutional provision on the subject.

If this legal problem is solved along historical lines and if the Courts are finally convinced that the tremendous street-widening problems of this country are problems of sovereignty, and if the decisions are quite uniformly announced to that effect, then the great financial problem that is involved is also solved.

In other words, if the Courts would uniformly hold that the great American problem of widening streets to take care of present and future traffic needs is a problem of sovereignty. Then as a matter of sovereignty the streets and highways can be widened the same as buildings can be destroyed to prevent the spread of fire or the same as a high-salaried man can be quarantined to prevent the spread of disease. All these things are done under a recognized governmental power which is older than any Constitution. Under such circumstances the property that it is necessary to acquire, has not been taken for a public use under any clause of any State or Federal Constitution and it is not neeessary to pay compensation to the abutting owner.

The mere statement of the proposition creates resistance and the writer proposes not only to show how historically sound it is; but to show how close some of the Courts are to a correct solution of the problem at present.

No space will be devoted to a statement of the necessity of widening streets and highways throughout this broad land. Engineers are more familiar with that phase of the question than the writer, who will just take it for granted. The enormous financial problem involved also requires no further discussion other than a re-statement of the fact that the correct solution of the legal problem at the same time solves the financial problem. Of course, the financial and legal problem is twofold because now it is not only necessary to re-build the old "horse-and-buggy" streets into automobile highways, but it is also necessary to plan for future requirements, along the streets which are being laid out, with the tremendous growth of self-propelled vehicles in mind.

The Engineer in Relation to Legislation Concerning Set-Back Lines

Without any reflection on engineers the writer knows from personal talks with them that quite generally they are not familiar with the Constitution of the State in which they practice. Ordinarily, the engineer realizes somewhat vaguely that there is a clause in his State Constitution as well as in the Fed-

eral Constitution to the effect that private property shall not be taken for a public use without compensation. The engineer is quite thoroughly convinced that the widening of a street and the establishment of a set-back line involves the taking of property from an abutting owner for a public use and, for that reason, compensation is necessary. Then he gasps at the billions and millions of dollars that are required to do what ought to be done to the present "horse-and-buggy" streets in order to convert them into automobile highways.

Engineers can do a great service by compiling and broadcasting throughout the land an estimate of the billions of dollars which it would take to put the highways of the United States in a safe condition for automobile traffic. A few years ago the writer drew the first ordinance for regulating the heights of buildings in Milwaukee, Wis. Twenty years previously the regulation of building heights in this country was practically unknown. Legislators were still adhering to the old principle that a man had the right to do with his own property what he pleased as long as he did not reach over on to his neighbor's property and commit some active trespass. The Courts had already recognized that he might reach over actively either by creating offensive odors or a nerve-racking noise; but up to the time of the Milwaukee ordinance no one had admitted that the regulation of building heights was a function of sovereignty. At about that time American engineers offered the statement that there were enough people on Lower Manhattan Island at ten o'clock in the morning to fill the streets to the top of the fourth floor windows if, for any reason, the people should rush into the street at the same moment. That statement startled the country and the Courts began to realize that the regulation of building heights was in fact a function of sovereignty. Since then they have held that the municipal units have the right to say that building heights shall only reach certain limits for the simple purpose of preventing more people from getting into a district than can be accommodated in the streets.

What the writer wishes to emphasize is that if the engineering societies of this country will inform the Courts of the billions of dollars that are required at the present time to put the "horse-and-buggy" streets in a safe condition for automobile traffic and, at the same time, tell them of the tremendous loss of life and damage to property that is constantly taking place due to the fact that such streets are not safe for automobile traffic, these organizations will thereby do much to cause the Courts to realize that the solution of this great traffic problem is a function of sovereignty.

There is nothing new or novel in this suggestion. The writer does not intend to do any violence to the clause in any Constitution which, in round phrases, states that private property shall not be taken for a public use without compensation. All he suggests is that the Courts, through the engineers, should be made to realize that the great traffic problem of this country to-day is as much a problem of sovereignty as fire-fighting, quarantine to prevent the spread of disease, the peremptory destruction of nuisances, entry upon private land to construct levees to prevent great rivers from destroying property, and the various other phases of recognized sovereignty.

The writer knows from personal experience that engineers and city planners have had more to do with the establishment of the legality of the building

heights regulations and zoning laws than the legal profession—in that engineers and city planners furnished the facts, figures, etc., and, in some instances, even submitted non-technical briefs to the Courts. Furthermore, engineers can do more than lawyers in convincing the Courts that the present tremendous traffic problem (which is at the bottom of the requirement of set-back lines to street widening) is a phase of sovereignty.

The lawyer will simply declare that it is a struggle between the police power and the Constitutions of the various States. The term, "police power", is a very unfortunate one. It suggests bluecoats, brass buttons, and billy-sticks, or something of that kind, when, in fact, it means the power of government or sovereignty, or the power of a people to preserve itself.

Private Property for Public Use Without Compensation

Mere words are often misleading unless the reader is able to go back of them, into history, and know the fundamental principles. Consider the Federal Constitution and the Constitutions of all the States. They contain a composite provision that private property shall not be taken for public use without just compensation. The word, "property", offers no difficulty. Some difficulty may be encountered with the word, "taken", but it is necessary to trace the meanings even farther back. Every one knows that even before the establishment of any American colony, Governments were taking property as a phase of sovereignty without compensation, and they are still doing it. For illustration may be cited: Quarantine; the abatement of public nuisances; the destruction of contaminated food; the destruction of valuable buildings to prevent the spread of fire; the confiscation of gambling devices; and the peremptory right to enter upon private land and repair levees, which is quite common throughout the Southern States.

Many other illustrations of sovereignty might be cited, most of which are older in principle than any Constitution in this country. These powers were extended to the very existence of sovereignty in its simplest form. They were all recognized before America was discovered as being phases of the power of Government. When the Fifth Amendment to the Federal Constitution was written, which stated plainly that private property shall not be taken for a public use without compensation, no one had in mind the prevention or stopping of the further exercise of this recognized phase of sovereignty. The Constitution of the State of Wisconsin was written in 1848. Section 13 of Article 1 reads as follows: "The property of no person shall be taken for public use without just compensation therefor." Every one will admit that in those words there was no intention to abolish quarantine; there was no intention to prevent the destruction of buildings to stop the spread of fire; and there was no intention to abolish the power of abating nuisances. The Wisconsin Courts have recently ruled that the sovereignty of the State may say to a man that he may not put a certain kind of a building in a residence district (Carter *vs.* Harper, 182 Wis. 148); that he cannot construct his building beyond a certain height in a certain district; and that he cannot build within 15 ft. of the street on either side of a corner lot (Hayes *vs.* Hoffman, 192 Wis. 63).

If a person can be quarantined thus causing him to lose his salary without compensation, or if a Government can destroy his property to prevent the spread of fire, or if his land can be seized without compensation for the purpose of constructing a levee to prevent the spread of a river, then it should be possible to convince the Courts that the United States is confronted with a very dangerous traffic problem, the solution of which is necessary to existence. After that it is not a long step for the Courts to state that this traffic problem can be solved as a phase of sovereignty, and that the acquisition of the necessary property is not a prohibited taking within the constitutional provisions, any more than the other confiscations.

The Courts have already held, and are now uniformly holding that, as a phase of so-called police power or as an element of sovereignty, a State may prevent a property owner from building above a certain line or beyond a certain distance from the street line. It is not asking them to go much farther to require that they recognize that the present traffic problems and the necessity for street widening are indeed phases of sovereignty, and that the execution thereof can be carried out without compensation, the same as quarantine or the destruction of property for fire-fighting purposes.

Largely because engineers expect the law to be an exact science, they look upon the growth of such problems in the Courts with suspicion. They find certain words in a Constitution and expect to go out and erect a series of legal decisions with those words in the same manner as they use a plumb-bob to erect a perpendicular. The writer would urge that engineers be patient with the Courts and not lose their respect for them even if they do find them about-facing in a proposition of this kind. It is only natural that twenty-five years ago judges did not state that the automobile traffic was a problem of such seriousness that it was a phase of sovereignty, because then it was not such a problem. Is it now not a fact that the problem has become so nation-wide, that the loss of life and destruction of property are so tremendous, and that the present effort to use the old-style streets for self-propelled vehicles is creating so many difficulties and hazards that, in fact, the problem has become one of sovereignty? In widening streets and providing for the solution of the problem, the power of government must support the undertaking as, in the past, it has supported the effort to combat disease and the spread of fire.

Citation of Legal Decisions in Wisconsin

In the case of Hayes *vs.* Hoffman (192 Wis. 63), decided in 1927, the validity of the set-back line was established under a zoning ordinance. There is only a short step from that zoning set-back line to one that would result in the widening of those very streets for traffic purposes. The defendant owned a vacant lot on the northwest corner of a street intersection and the plaintiff was a neighbor across the street on the southwest corner. At that time, Milwaukee had a zoning ordinance which had not yet been interpreted by the Courts; it required a 15-ft. set-back line in that residence district. City officials had construed the ordinance to mean that the 15-ft. set-back line was applicable only to the front of the property and not to the side; in other words, that it would be applicable in that case on the street passing between the two

properties. The defendant began building operations under that interpretation of the law. The plaintiff lived across the street and at the end of 31st Street so that if the defendant built within the 15-ft. line along the side street he would obstruct the view of the plaintiff. The latter enjoined the building operations on the ground that, by building to the 15-ft. line along the side street, the defendant obstructed his view, and the Supreme Court held that the 15-ft. set-back line applied to both sides. This left the defendant a piece of ground that was too small for building anything of value, but the City of Milwaukee paid him nothing, because it was established in Wisconsin that the zoning ordinance is an element of sovereignty.

In other words, it was enacted under the so-called misnomer, "police power". If, in this case, the defendant was unable to build on this 15-ft. strip on either side, has not his property been taken? And if that restriction is placed there for air, light, and sanitation, or for any other reason which can be written into a zoning ordinance, then was not his property taken for a public purpose? Nevertheless, there was no compensation, because the so-called "taking" was under a phase of sovereignty, which is older than any Constitution of this land. The defendant in this case was merely in the same position in which he would have been if buildings on these 15-ft. strips had been dynamited to prevent the spread of fire. He had no more claim against the public for compensation than if he had been quarantined away from a $25 000 job for two months to prevent the spread of smallpox.

This case can be used to define a line of demarcation between "the taking for a public use" as mentioned in the various Constitutions which requires compensation, and "the taking" in the case of Hayes *vs.* Hoffman which was undeniably a confiscation, but which comes under the head of sovereignty and does not require compensation. It is admitted that the owner cannot put any buildings on either one of the 15-ft. strips. He can run a lawn mower over both of them. He might have flower beds in them, but he cannot even build a dog house on either one of them. If the City permits automobiles to run over either one of these 15-ft. strips, the City apparently would have to pay the owner if it was done under any municipal recognition, although if automobiles gradually began driving over both the strips and kept it up for twenty years, the highway will be established without compensation. If the City should go in and run a lawn mower over these 15-ft. strips, or put in flower beds, there might be a claim that the City had made a park out of them and for that reason would have to pay compensation. The question is that narrow. The City in this very case prevents the owner from building over these 15-ft. strips so that air currents can pass up and down, so rays of sunlight may pass in, and so that the view of the street is improved; and this is all done without compensation. It is a short step to the claim that the city may permit automobiles to pass over the 15-ft. strips the same as light rays or air currents. The automobile traffic in either of the streets may be so tremendous that the City, in the exercise of its sovereignty, may class the problem as one that is just as dangerous to the people as any of the others.

Several years ago the Wisconsin Legislature was bold enough to state in one of these laws[3] that a set-back line as a notice of future street widening could be established without any compensation to the abutting owner.

The set-back line may be established without compensation; its establishment is not considered equivalent to the taking of property, although it does provide that the owner who builds over the line does so at his peril and cannot recover any damages for the building beyond the line. The validity of this statute has not been litigated in the Courts although it has been on the statute books for several years. The establishment of the set-back line has many elements of fairness, and it is not as harsh as quarantine, the destruction of buildings to prevent the spread of fire, or the destruction of farms to prevent the spread of a river. Another law recognizes the necessity of paying compensation to the abutting owner because the other law was drafted a great many years ago as a part of the charter of the City of Milwaukee, and in that age when the seriousness of the automobile traffic problem had not yet been realized.

The history of the world shows that sovereignty knows no limits. When put to the ultimate test it recognizes no property rights. The self-preservation of a people is a law unto itself. A written constitution is an agreement between the then existing units of people who make it a sovereignty concerning the exercise of the phases of government mentioned therein, and there is not in existence a single constitution which contains an abdication of all phases of sovereignty not mentioned therein.

A set-back line which serves to notify a property owner that within a reasonable time in the future it is apparent to the authorities that the governmental sovereignty will be called upon to expand a street for the solution of this great threatening problem, is much fairer to the property owner than the sudden destruction of his building to prevent the spread of fire or the peremptory and sudden quarantine of his body to prevent the spread of disease. The cry that generally goes up in the United States is that this is the richest country in the world and that the Government should pay and pay well for these strips of land necessary to convert the out-of-date streets into automobile traffic highways. This country may be the richest nation in the world; it may have the richest group of States in the world; but this problem is not a National one, although it is nation-wide, and it is not a State problem, although it exists in every State in the Union. It is a problem that must be financed ordinarily by the local municipal units, such as towns, cities, and villages, and it is a well-known fact that such municipal units are habitually financially embarrassed.

The writer has had some experiences with street widening that have changed his views considerably concerning the harshness that might be apparent at the first mention of the necessity of taking over this problem as a phase of sovereignty. In Milwaukee there have been many instances in which the remnants of property left over after street widening, were worth several times the original value of the entire lot before the widennig. Admittedly, there will be cases of hardship when the Courts are brought to realize that the necessity

[3] The Wisconsin statute which claims that the set-back line can be established without any compensation to the abutting owner is Sub-Divisions (10) and (11) of Section 62.23, of the General Statutes.

of street widening is a case of sovereignty (which means it will be done without compensation for the land taken); but there are hardships in every phase of the contact between the private citizen and the administration of public duties. In his twenty years of experience in municipal law the writer has been forced to recognize that even taxes are unequal; that the jury duty burden does not fall equally upon the public; and that the support of the public schools cannot be made equal. In fact, not one single burden in the public field can be said to fall equally upon the people. The statement that there may be hardships in the proposed solution of the problem does not condemn that solution. For each case of apparent injustice and hardship where there may be a set-back line in a necessary widening of the street without compensation to the abutting owner, there will be dozens of cases in which the widening of the street will create enormous additional property values. The advantage of the set-back line is that it can be so slowly and gradually established as a street line as to remove most of the injustice. It might even be provided in such a law that, if there must be a "taking" within five years of the establishment of the set-back line, there will be compensation; but that there should be no compensation after a 5-year notice of the establishment of the line. On some streets the advance notice might even be extended to ten years.

Cases of the Exercise of Governmental Sovereignty

Suppose the City of Milwaukee should serve notice on an owner that five years from date his house would be blown up to prevent the spread of a dangerous fire which it is known would happen. That would give the owner plenty of time to move and re-adjust himself. And, again, suppose the City of Milwaukee should say to a man that he would be quarantined in five years for three weeks to prevent the spread of a smallpox epidemic which the municipal authorities know is going to occur. The comparison does not need to be followed further. If it is recognized that the correct solution of this great traffic problem is a phase of sovereignty, then this set-back law provides for instrumentalities for removing the hardships which are incident to the very cases in which the Government does exercise its sovereignty without any compensation whatever.

As the United States continues to grow and its problems become more complex, a great many of the old ideas which have been instilled into the minds of its people will have to be discarded. People seem to like to say that "a man's house is his castle". They seem to enjoy saying that this is a country of liberty and a man has absolute ownership of his property. What is ownership? Nothing is more mythical than a title. Title is nothing but a gift of sovereignty. If title is claimed to a piece of real estate in America to-day, what is the nature of that claim? It is merely an evidence that some one took the land away from the Indians as a matter of military strength, or by some other method of that kind, and that as a matter of sovereignty there is, at the beginning of the chain of title, a patent from the Government. That title is no stronger than the sovereignty under which citizens are living.

The very existence of the title depends on sovereignty. In fact, the title is simply an instrument of sovereignty, because the Government with its Courts

is what makes the title to land good, so that, after all, there is not much hardship in permitting sovereignty to deal with that title or to restrict it so as to comply with the requirements that spring up in the self-preservation of a great people. For instance, consider the statement that people like to "roll under their tongues" to the effect that "a man's house is his castle". When two castles are built close together things begin to happen. For instance, one castle owner cannot dig down on the property line so as to disturb and cause the soil of the neighboring castle owner to fall in. As a result the easement of lateral support has grown up. One castle owner cannot run a tannery on his property so as to offend the nose of his neighbor. Things gradually developed so that he was not allowed to create other nuisances, such as disturbing noise. When a castle owner lived all alone he could throw his garbage out on the surface of the ground, but when other castles commenced to be built in the neighborhood he found that his so-called liberties were being restricted. As the castles continued to spring up, their owners found that they could not build so close to the property line as to shut out light and air; and in the great cities buildings cannot be constructed so as to attract more people to the district than can be accommodated safely. It is not going very far to state that, since it is recognized that a castle owner must give up many of his so-called rights, such as in the case of zoning and building height restrictions, he will likewise have to give up land so that streets may be expanded to solve one of the greatest problems that has ever confronted any country.

Conclusion

American Courts will soon hold uniformly that the solution to the traffic problem is a phase of sovereignty and that private rights shall give way, as they must, to fire-fighting, abatement of nuisances, prevention of crime, the operation of levees, and to those other forms of sovereignty that are happening every day. When that time comes the public will accept it the same as they accepted zoning, building-heights regulations, billboard ordinances, health regulations, ordinances confiscating contaminated food, laws confiscating gambling devices, laws permitting the restraining of floods, and other recognized phases which are accepted without question and without any compensation whatever to the people who are being called upon to submit to the exercise of the sovereignty.

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Founded November 5, 1852

DISCUSSIONS

REPORT OF THE COMMITTEE OF THE IRRIGATION DIVISION ON "A NATIONAL RECLAMATION POLICY"

Discussion

BY THE COMMITTEE

COMMITTEE OF THE IRRIGATION DIVISION ON "A NATIONAL RECLAMATION POLICY" (by letter).[1]—On November 1, 1926, the Irrigation Division appointed a committee composed of eleven Western irrigation engineers to consider and report upon "A National Reclamation Policy". After much correspondence two meetings of the Committee were held, and a unanimous report was finally adopted in San Francisco, Calif., in July, 1928. The report of the Committee was presented to the Division at San Diego, Calif., in October, 1928, and discussed in two papers[2] presented by Elwood Mead, M. Am. Soc. C. E., and by the Chairman of the Committee, J. B. Lippincott, M. Am. Soc. C. E.

Twenty-four discussions have been presented during the two years after the publication of this report in September, 1928, by engineers of the United States, Canada, Australia, and India. The Committee devoted more than a year's time to a consideration of its report which was unanimously signed before its presentation to the Irrigation Division. The Committee now has reviewed its original report, and the discussions, and does not find occasion to modify or amend its judgment of July, 1928.

President Hoover presented a letter, under date of August 21, 1929, to the Conference of Western Governors and Representatives, held in Salt Lake City, Utah, on August 27, 1929, relative to the control of public lands, minerals, and reclamation in the West. The parts of this letter that bear on the policy of Federal Reclamation are, as follows:

NOTE.—Committee's closure. Discussions of this report have appeared in *Proceedings*, during successive months, as follows: August, 1929, by Messrs. George W. Fuller, W. F. Russell, Nils A. Olsen, and L. C. Gray; September, 1929, by Messrs. Frank Adams, Munson J. Dowd, C. E. Grunsky, S. T. Harding, and H. H. Dare; October, 1929, by Messrs. Charles R. Hedke, Charles Kirby Fox, Rhodes E. Rule, and George M. Bacon; November, 1929, by Messrs. David Weeks, R. K. Tiffany, and G. E. P. Smith; December, 1929, Messrs. John A. Widtsoe and F. H. Newell; February, 1930, by Messrs. A. D. Lewis and Joseph Jacobs; April, 1930, by Messrs. F. N. Cronholm, R. I. Meeker, and E. B. Debler; and September, 1930, by John E. Field, M. Am. Soc. C. E.

[1] Received by the Secretary, December 4, 1930.

[2] *Proceedings*, Am. Soc. C. E., May, 1929, Papers and Discussions, pp. 1193 and 1207.

"It may be stated at once that our Western States have long since passed from their swaddling clothes and are to-day more competent to manage much of these affairs than is the Federal Government. Moreover, we must seek every opportunity to retard the expansion of Federal bureaucracy and to place our communities in control of their own destinies. The problems are in large degree administrative in character, both as they affect the Federal Government and the Government of the States."

As to the Reclamation Service, President Hoover continued:

"It seems to me that the vital questions here are to re-orient the direction of the Reclamation Service primarily to the storage of water and to simplify its administration.

* * * * * * * *

"The present Reclamation Act is based fundamentally on the reclamation of Government-owned lands. Possible areas available for reclamation have now passed almost wholly into private ownership and the use of the Reclamation Fund for further projects may be legally criticized owing to the fact that the land is no longer part of the public domain and circumlocution by voluntary agreements may not always be possible.

"Moreover, the application of the fund under the present organization results in very large Federal administrative activities within the States of a character which was never originally contemplated and which could be much better administered by the local State Governments themselves. In many ways it duplicates the State water administrations."

The position of President Hoover in connection with the policy of Federal Reclamation is of particular interest because of the many respects in which he presents similar opinions to those of the Committee.

The Committee calls attention to Principle (6) of its original report, which reads:

"(6) The output of power and of water at Federal or State works should be disposed of at wholesale and not at retail."

The intention of the Committee was to express its opposition to either the State or Federal Government entering into the distribution of either water or power to individual consumers thereof. Its belief was that the actual contact with such consumers should be through local organizations, such as districts, mutual water companies, or private corporations. The Committee recognizes that, as provided in the Reclamation Act, the control and distribution of the water are vested in the respective States.

The Committee endorses as a fundamental policy the proposal that the Federal Government should expend the money derived from the sale of public lands and the natural resources derived therefrom, in the arid land States, in the improvement and development of those States by the conservation and utilization of their water supplies. The Federal and State Governments should co-operate in the construction particularly of storage reservoirs for irrigation, flood control, and other water supply projects. The province of these reservoirs is to regulate the stream flow in such a manner as to make them of public benefit to all classes of water users irrespective of whether they are on Federal Reclamation Projects, or otherwise. This policy of development of

the West is of great public importance not only to the Western States themselves, but to the Nation as a whole.

(Signed)

J. B. LIPPINCOTT, *Chairman*,
M. C. HINDERLIDER,
R. K. TIFFANY,
G. E. P. SMITH,
W. H. CODE,
JOHN E. FIELD,
GEORGE M. BACON,
JOSEPH A. ELLIOTT,
J. C. STEVENS,
BURTON SMITH (No expression of approval or disapproval received.)
C. E. GRUNSKY ("Without approval of the first sentence of the last paragraph because appropriations should be for specific aid and should not be made indirectly in unknown, uncertain amounts.")

Committee.

October 30, 1930.

The original report of the Committee[4] is appended, as follows, for purposes of convenience and reference:

REPORT

By resolution the Committee limited its work to a study of policies governing the reclamation of arid lands and related matters.

The policy of the United States and of the several States in the matter of water conservation and arid land reclamation should be controlled by certain basic principles as follows:

(1) The waiving of interest payments to land owners on Government reclamation projects is unwise. In the future Government contributions should appear in the assumption of a part of the cost of project works and not in the granting of relief to the individual farmer such as the waiving of interest charges.

The United States Bureau of Reclamation has formulated a program of construction, covering the ensuing ten years, involving expenditures of approximately $100 000 000. To the extent that commitments have been made the Bureau should fulfill its assumed obligations and, on the other hand, the land owner should be required to meet his obligations or surrender his holding in the Government project.

[4] *Proceedings*, Am. Soc. C. E., September, 1928, Papers and Discussions, p. 2098.

(2) The conservation of the water in the rivers and lakes of the country should be under public control; and in order to lay a proper foundation for the making of comprehensive plans the Federal and State Governments should gather data, compile statistics, and conduct studies necessary to determine the feasibility of projects.

(3) The regulation of the flow of streams for the prevention of floods and for the best possible utilization of the waters should be undertaken by the States, or jointly by the United States and the States under such suitable forms of co-operation as may be appropriate under the Constitutional authority now delegated to each. They should prepare and adopt comprehensive plans for such regulation and should bear an equitable portion of the cost of water storage and flood-control work when the economic aspects after full investigations are found to be favorable, and the remainder of the cost should be allocated to flood control, irrigation, power development, municipal water supply, and other purposes.

(4) Where protection against flood waters results from the regulation of stream flow by means of reservoirs or otherwise, the proportion of the cost of the flood-control work not assumed by the Federal or State Government should be assessed against the lands and other properties which receive benefit therefrom.

(5) Municipalities or other public agencies or private parties should be allowed to construct approved projects in conformity with the approved plans; subject, however, to the public control of reservoirs and to the recapture after a reasonable time by the public of any franchise or similar rights conferred on private parties.

(6) The output of power and of water at Federal or State works should be disposed of at wholesale and not at retail.

(7) Interested States under suitable interstate compacts should be permitted by the United States to undertake the regulation of interstate streams.

(8) In the carrying out of further stream-regulation work, preference should be given to the construction of regulating reservoirs and the development of supplemental water supplies for existing irrigation systems, whether Federal or otherwise.

(9) Agricultural conditions due to over-production are such at present that it is undesirable for the Federal Government, except in the case of commitments already made, to bring new areas under cultivation.

(10) The construction of new irrigation projects should not be authorized except after thorough investigation and favorable recommendation by a Board of Review. This Board should include competent construction engineers, engineers with special operating and agricultural experience, economists, and financiers familiar with local production and marketing conditions. The State should share in the responsibility for the selection and approval of projects. In determining the feasibility of proposed projects, State lines, local interests, and political expediency should not control.

(11) When new projects are authorized, principal and interest payments on construction costs should be required of the land owner. The interest

rate should be low and the principal payments extended over a long period, with no payments on principal during the early years.

(12) The plan of repayment of construction costs of reclamation should be put into operation on each unit of the project at an early date after completion. The plan of payment should be sufficiently elastic to meet the settlers' ability to pay, but no relinquishment in the terms of repayment, once they are put into effect, should be permitted.

(13) In the case of reclamation projects, it should be recognized that settlement of the land is fully as imperative to success as construction. It can be greatly stimulated by the Government or other authorities taking drastic measures to prevent land speculation. The Department of the Interior is to be congratulated on its efforts to curb land speculation in recently authorized projects.

(14) Land settlement including paternalistic, financial, or any other kind of assistance to the individual farmer should be treated as a local matter and should therefore be made the concern of the State or locality rather than of the United States. Aid extended by Federal Land Banks has been generally helpful and the possibility of further extension of such aid by this or similar agencies is worthy of serious consideration.

(15) The Federal Government should continue its present policy of relinquishing control of completed works to suitably organized local agencies as soon as practicable.

(16) According to the report of the U. S. Bureau of Reclamation for 1926, the total area of land provided with water for irrigation was 1 803 000 acres in reclamation projects, of which, 1 320 000 acres were being cultivated by irrigation—a record of which the country may well be proud.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

RELATION BETWEEN RAIL AND WATERWAY TRANSPORTATION: A SYMPOSIUM

Discussion

BY C. W. KUTZ, M. AM. SOC. C. E.

C. W. KUTZ,[1] M. AM. SOC. C. E. (by letter).[1a]—In 1925, the writer prepared a paper on the "Relation of the Ohio River and Its Tributaries to Transportation in the United States."[2] At that time, the 9-ft. project was incomplete, and the tonnage moving on the Ohio River itself, exclusive of tributaries, was 15 725 849. To determine the water-haulage cost of moving this tonnage and its relation to published rail rates on like commodities that have the same points of origin and destination, a comprehensive analysis was made of the 1925 tonnage.[3]

Nineteen separate movements were studied and for each is shown the rail rate and the water-haulage cost as secured from the principal users of the river. Some of these users were common carriers and some contract carriers, but the large majority was private carriers. In the first two classes the water-haulage cost included an item of profit to the owner. In the third class the water-haulage cost included a 6% return on the investment in floating plant and on the privately owned terminal facilities, a return which is comparable to that which the railroads of the United States are permitted, under the Transportation Act, to earn on their investments.

As most water movements involve higher terminal charges than corresponding rail movements, the difference between the rail rates and the water-haulage costs was reduced by the estimated terminal differential. This method of determining economic justification is superior to a comparison of cost per ton-mile, as it eliminates differences due to the generally greater length of the water route, and it also avoids the faulty comparison of a long-haul rail

NOTE.—Symposium published in March, 1930, *Proceedings*. Discussion of this Symposium has appeared in *Proceedings*, as follows: April, 1930, by Lewis C. Sorrell, Esq.; May, 1930, by Messrs. R. B. Kittredge, Harry A. Palmer, and J. Parker Snow; August, 1930, by Messrs. Baxter L. Brown and Theodore Brent; September, 1930, by Samuel O. Dunn, Esq.; and December, 1930, by Henry W. Hobbs, M. Am. Soc. C. E.

[1] Brig.-Gen., U. S. A. (*Retired*), Washington, D. C.

[1a] Received by the Secretary, November 3, 1930.

[2] *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 1105.

[3] *Loc. cit.*, Table 23, p. 1178.

rate with a short-haul water rate. In 1925, the average length of haul on railways was in excess of 200 miles, whereas that on the Ohio River was less than 90 miles.

It may be noted that this table (Table 23[4]) dealt only with the Ohio River, excluding all its tributaries, some of which, such as the Monongahela River, are enormously profitable, while others are economic failures. When the movement involved the use of a tributary, as well as the Ohio River, the difference between the rail rate and the water-haulage cost was prorated between the two streams on the basis of the mileage of each involved in the movement. The table showed that the differences between rail rates and water-haulage costs, when applied to the 1925 tonnage, resulted in a saving to the shippers for that year of $5 328 594, which was about 87% of the annual charges for 1925.

The project was completed in 1929 at a capital cost of about $115 000 000. The annual cost of operation and maintenance was then estimated at $2 350 000. If interest at 4% and a depreciation or obsolescence factor based on an assumed life of 50 years (sinking-fund basis, 4%), is added, the annual charges of the completed project will be equal to $7 703 000.

TABLE 9.—Comparisons Between Tonnage, Ton-Mileage, and Savings.

Year	Tonnage	Ton Mileage	Savings
1925	15 725 849	849 195 846	$5 328 894
1926	19 754 978	1 101 629 085	6 911 600
1927	20 128 518	1 214 490 565	7 098 600
1928	20 938 267	1 344 687 000	7 796 000
1929	21 955 000	1 512 584 000	8 174 106*

* Estimated, increase being proportioned to the increase in tonnage.

Table 9 shows how the commerce and the savings, based on similar analyses, have increased since 1925. From this table and from the estimated annual charges, it appears that the canalized Ohio River, although just completed, is already yielding savings to shippers in excess of the annual charges. As its capacity is many times greater than the present tonnage, the economic justification of the investment seems to be assured.

[4] *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 1178.

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DISCUSSIONS

THE SHANNON POWER DEVELOPMENT IN THE IRISH FREE STATE

Discussion

BY MESSRS. L. J. BEVAN AND A. R. C. MARKL

L. J. BEVAN,[1] M. AM. SOC. C. E. (by letter).[1a]—This paper presents in excellent form an interesting description of a brave project. From a power point of view Ireland is in an undeveloped condition similar to that of the United States about 1900. It has a total installation of 27 000 kw., with an urban population of 1 000 000, or 0.027 kw. per inhabitant. The annual consumption of 40 000 000 kw-hr. in 1920 when the Shannon Development was being proposed was 40 kw-hr. per inhabitant, as compared with 71 kw-hr. for all Europe and 472 kw-hr. for the United States. Economically, the Shannon Project should do much for the prosperity of Ireland.

This was referred to as a brave project primarily because of the large amount of power contemplated—500 000 000 kw-hr.—from 37 in. of rainfall on 4 000 sq. miles of drainage area, through a 100-ft. fall. At 80% efficiency from water to bus-bar 530 cu. ft. of water are required per kilowatt-hour. The output contemplated from the first development is 150 000 000 kw-hr., which corresponds to an average annual flow of 2 520 cu. ft. per sec., or a run-off-rainfall ratio of 23 per cent. This is reasonably to be expected.

The ultimate development, 500 000 000 kw-hr., similarly requires an average annual flow of 8 520 sec-ft., or a run-off-rainfall ratio of 77 per cent. Such a run-off ratio is so large that a further elaboration of this phase of the project would be of intense interest.

A. R. C. MARKL,[2] JUN. AM. SOC. C. E. (by letter).[2a]—The general description of the development and its basis was curtailed by the writer to a minimum, so as to provide more space for the explanation of a novel method of handling earthwork, which formed the principal object of the paper. Mr. Bevan's questions demand an elaboration which would seem to be beyond its original purpose. Perhaps the following figures may serve as a reply.

NOTE.—The paper by A. R. C. Markl, Jun. Am. Soc. C. E., was published in April, 1930, *Proceedings*. Author's closure.

[1] Cons. Engr., New York, N. Y.

[1a] Received by the Secretary, April 25, 1930.

[2] Civ. Engr., Arthur G. McKee & Co., Cleveland, Ohio.

[2a] Received by the Secretary, November 25, 1930.

In the 4 011-sq. mile area from which the Shannon River draws its water, an average annual rainfall of 37.25 in. obtains, corresponding to a total of 347 436 000 000 cu. ft. per year. Due to a signal lack of forests and the rockiness of the ground, 75 to 78% of this amount, that is, 270 156 000 000 cu. ft. per year, or 8 475 sec-ft., flow off. With a head varying between 93.5 and 110.7 ft., dependent on the tides, which travel up the tail-race, and an estimated rate of 584.8 cu. ft. per kw-hr., this produces 462 000 000 kw-hr. at the bus-bars.

The maximum flow occurs during the winter, while in the months from May to September rarely much more than 3 000 cu. ft. per sec. are carried by the river. This distribution, of course, is fortunate inasmuch as it produces a coincidence of supply and demand. An effective storage capacity of 20 235 000 000 cu. ft., which serves to achieve a complete balance of the water household, is obtained by providing for water-level variations of between 4.4 and 12.0 ft. at the three principal lakes through which the Shannon River flows. These lakes, Loughs Allen, Ree, and Derg, cover a total area of 101 sq. miles.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

ANALYSIS OF CONTINUOUS FRAMES BY DISTRIBUTING FIXED-END MOMENTS

Discussion

BY MESSRS. JENS EGEDE NIELSEN, F. E. RICHART, AND WILLIAM A. OLIVER

JENS EGEDE NIELSEN,[1] ASSOC. M. AM. SOC. C. E. (by letter).[1a]—The large number of papers dealing with moment distribution in continuous beams and rigid frames is proof enough that this question needs a practical solution.

The method, as outlined by Professor Cross, is so easily memorized and so quickly applied that all approximate methods and guesswork profitably can be eliminated without the sacrifice of speed, which often is as essential as economy.

When balancing the moments at a point such as C, Fig. 1, in a system in which the signs are quite arbitrarily considered positive when causing sags downward in horizontal members and toward the right side of vertical members, it is not entirely evident which is the right and which the left side of the joint or, rather, which moments should be added and which should be subtracted in order to determine the unbalanced moment. If, however, it is borne in mind that moments in members which sag clockwise should balance moments in members which sag counter-clockwise, it is easy to visualize which moments should be added and which subtracted. Thus, in Fig. 24, -200, -50, -100, which members sag counter-clockwise, should be subtracted from -400, which sags clockwise, giving an unbalanced moment of $-400-(-200-50-100) = -50$ to distribute among the four members.

The determination of the fixed-end moments for different loadings can be greatly facilitated by the use of influence lines, such as those given in textbooks, one for concentrated loads and an integrated curve for uniformly distributed loads.

The limitations of the method should be realized at once. Systems in which shears are dominant (wind-bracing bents in tall buildings) and systems in which the longitudinal stresses are dominant (arches, etc.) and, particularly,

NOTE.—The paper by Hardy Cross, M. Am. Soc. C. E., was published in May, 1930, *Proceedings*. Discussion of the paper has appeared in *Proceedings*, as follows: September, 1930, by Messrs. C. P. Vetter, L. E. Grinter, S. S. Gorman, and E. E. Eremin; October, 1930, by Messrs. A. H. Finlay, R. F. Lyman, Jr., R. A. Caughey, Orrin H. Pilkey, and I. Oesterblom; and November, 1930, by Messrs. Edward J. Bednarski, S. N. Mitra, Robert A. Black, and H. E. Wessman.

[1] Cons. Engr. (Nielsen & George), Chicago, Ill.

[1a] Received by the Secretary, November 1, 1930.

systems in which moving loads are encountered, the time saving seems problematic, and other methods should be considered.

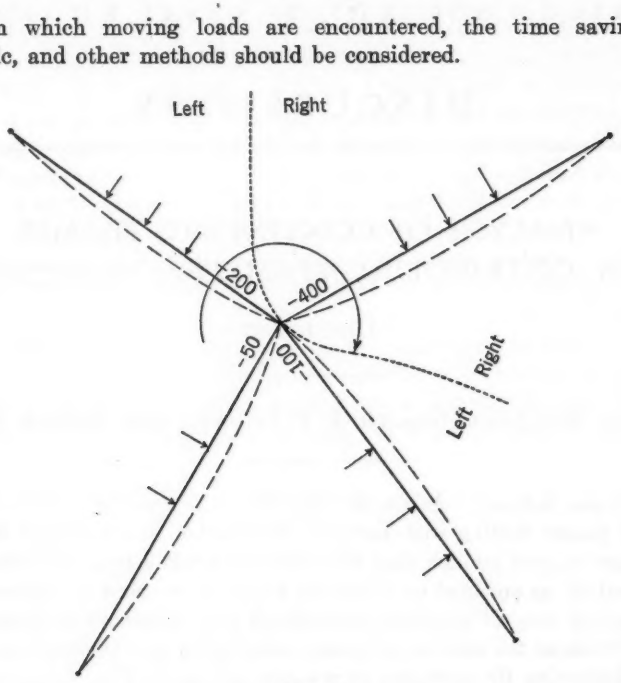


FIG. 24.

The paper is most valuable and should prove helpful to practicing structural engineers.

F. E. RICHART,[2] M. AM. SOC. C. E. (by letter).[2a]—The moment-distribution method of analyzing continuous frames originated by Professor Cross is one of the most important developments in this field in many years. The main advantage of the method is its simplicity, through which the designer can attain familiarity with a class of structures generally associated with long algebraic analyses. With this simplicity also comes rapidity of computation, which makes it possible for a structure to be analyzed under many different load conditions in the time previously taken for a single solution. In spite of its advantages, however, it will probably take designers in general a number of years to appreciate the method and to give it the widespread recognition that it deserves.

Because of a disinclination to confuse the reader's conception of the moment-distribution method, the writer hesitates to describe a variation which is nearly as simple in many cases and which leads directly to an exact solution. The scheme is based on an analysis by the slope-deflection method of various typical arrangements of flexural members, for which equations of a standard form may be written.

[2] Research Associate Prof., Theoretical and Applied Mechanics, Univ. of Illinois, Urbana, Ill.

[2a] Received by the Secretary, November 24, 1930.

In Fig. 25, the member, AB, meets the members, BC and BD, in a rigid joint at B. When a couple, M_{AB}, is applied at A (assuming there is no translation of the points, A, B, C, and D), the end, B, is partly restrained. The amount of restraint depends upon the stiffness, as measured by $K = \frac{I}{l}$ for each of the three members, and by the restraints at the points, C and D. In general, the moment, M_{AB}, is given by the equation,

$$M_{AB} = N_1 E K_1 \theta_A, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

in which, N_1 is a constant, having a value between 3 and 4; E is the modulus of elasticity of the material; K_1 is $\frac{I}{l}$ for the member, AB; and θ_A is the slope of the elastic curve at A, due to the action of the couple.

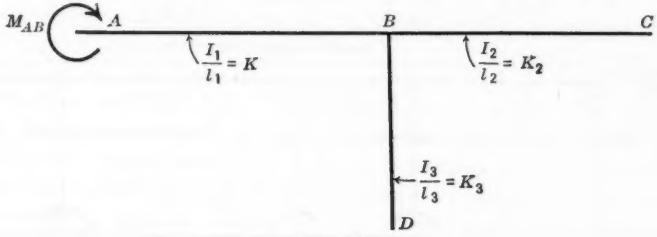


FIG. 25.

The value, N_1, may be written,

$$N_1 = 4\left(\frac{3K_1 + N_2K_2 + N_3K_3}{4K_1 + N_2K_2 + N_3K_3}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

N_2 and N_3 are constants similar to N_1; thus, if there is a hinge at C, $N_2 = 3$; if a fixed end, $N_2 = 4$; if the slopes at B and C are equal and opposite, $N_2 = 2$; and if the slopes are equal and alike, $N_2 = 6$. It will be noted that Equation (18) has terms in numerator and denominator which are alike except for the coefficients of K_1, which are always 3 and 4, respectively.

Another relation between the moments at A and B is as follows:

$$M_{BA} = \frac{M_{AB}}{2}\left(\frac{N_2K_2 + N_3K_3}{N_2K_2 + N_3K_3 + 3K_1}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

The moment, M_{BA}, is equal and opposite to the sum of M_{BC} and M_{BD}, and the latter moments are proportional to the respective values of N_2K_2 and N_3K_3.

The application of Equations (17), (18), and (19), to a continuous beam or frame may be started just as in Professor Cross' procedure. For a given loading, the "fixed-beam" moments are found and the unbalanced moment is determined at each joint. In Fig. 26, the values of K for the four spans are denoted by K_1, K_2, K_3, and K_4, and the unbalanced moments at B, C, and D, by U_B, U_C, and U_D. An exact method of distributing these unbalanced moments throughout the beam has been published.[3]

[3] "Analysis of Statically Indeterminate Structures by the Slope-Deflection Method," by Wilson, Richart, and Weiss, *Bulletin 108*, Eng. Experiment Station, Univ. of Illinois, 1918, pp. 133 and 137.

Consider the unbalanced moment, U_B, at B. This is distributed between the members, BA and BC, as follows:

$$M_{BA} = U_B \left(\frac{3\,K_1}{3\,K_1 + N_{23}\,K_2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

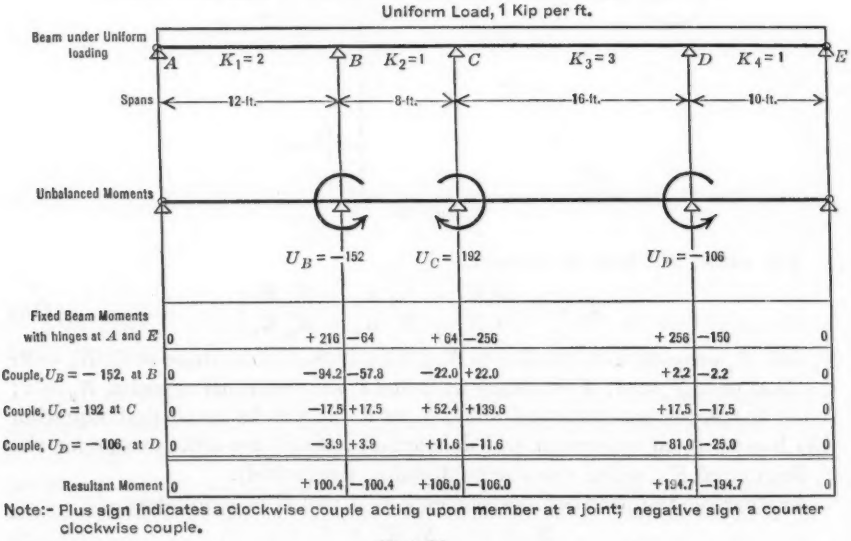
$$M_{BC} = U_B \left(\frac{N_{23}\,K_2}{3\,K_1 + N_{23}\,K_2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

$$M_{CB} = \frac{M_{BC}}{2}\left(\frac{N_{34}\,K_3}{N_{34}\,K_3 + 3\,K_2}\right) = -\,M_{CD} \ldots\ldots (22)$$

and,

$$M_{DC} = \frac{M_{CD}}{2}\left(\frac{3\,K_4}{3\,K_4 + 3\,K_3}\right) \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

in which, $N_{23} = 4\left(\frac{3\,K_2 + N_{34}\,K_3}{4\,K_2 + N_{34}\,K_3}\right)$ and $N_{34} = 4\left(\frac{3\,K_3 + 3\,K_4}{4\,K_3 + 3\,K_4}\right)$.



Note:- Plus sign indicates a clockwise couple acting upon member at a joint; negative sign a counter clockwise couple.

FIG. 26.

Similarly, at C, the unbalanced moment, U_C, is distributed, as follows:

$$M_{CB} = U_C \left(\frac{N_{21}\,K_2}{N_{21}\,K_2 + N_{34}\,K_3}\right) \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

$$M_{CD} = U_C \left(\frac{N_{34}\,K_3}{N_{21}\,K_2 + N_{34}\,K_3}\right) \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

$$M_{BC} = \frac{M_{CB}}{2}\left(\frac{3\,K_1}{3\,K_1 + 3\,K_2}\right) = -\,M_{BA} \ldots\ldots (26)$$

and,

$$M_{DC} = \frac{M_{CD}}{2}\left(\frac{3\,K_4}{3\,K_4 + 3\,K_3}\right) = -\,M_{DE} \ldots\ldots (27)$$

in which, $N_{21} = 4\left(\frac{3\,K_2 + 3\,K_1}{4\,K_2 + 3\,K_1}\right)$.

At D, the unbalanced moment, U_D, is distributed, as follows:

$$M_{DC} = U_D \left(\frac{N_{32} K_3}{N_{32} K_3 + 3 K_4} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

$$M_{DE} = U_D \left(\frac{3 K_4}{N_{32} K_3 + 3 K_4} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

and,

$$M_{CD} = \frac{M_{DC}}{2} \left(\frac{N_{21} K_2}{N_{21} K_2 + 3 K_3} \right) = - M_{CB} \quad \ldots\ldots\ldots\ldots (30)$$

in which, $N_{32} = 4 \left(\frac{3 K_3 + N_{21} K_2}{4 K_3 + N_{21} K_2} \right)$.

The first of the two subscripts used with N indicates the span nearest the origin of the moment being distributed.

Equations (17) to (30), inclusive, are given in full for further reference, but their significance can be expressed in a few words. An unbalanced moment, U, is distributed among the members meeting at the joint in proportion to their respective values of NK. This distribution is final. The value of the term, N, lies between 3 and 4 for the examples given, and the terms of the fraction have a typical form that is easily remembered. The moment at the adjacent joint is less than one-half the moment at the joint where the unbalanced moment is applied. These three types of equations are similar to the operations in Professor Cross' distribution and "carry-over" of moments, but the equations are applied only once to determine an exact final distribution of moments. However, it is necessary to distribute each unbalanced moment in turn, so that the final moments are the algebraic sum of the original "fixed-end" moments and the moments due to the three unbalanced couples.

As a numerical example, consider the four-span beam of Fig. 26 under a uniform load. With the ends, A and E, hinged, it is convenient to apply moments at B and D on the first and fourth spans, which apply to the case having one end hinged and the other fixed. Such "fixed-end" moments are three-halves as great as for the case in which both ends are fixed. The unbalanced moments at B, C, and D, are found to be —152, +192, and —106 units, and values of K are assumed at 2, 1, 3, and 1, respectively, for the four spans. The unbalanced moments are applied as couples producing a rotation in the direction indicated by the unbalance.

The values of the constants, N, are found to be $N_{21} = 3.6$; $N_{23} = 3.7$; $N_{32} = 3.2$; and $N_{34} = 3.2$. Using the convention as to signs followed in *Bulletin 108* previously referred to (a clockwise moment or attacking couple at the end of a member is considered positive), the moments at the ends of each span are given in Fig. 26 for the original fixed-end condition at B, C, and D, and for the successive application of the unbalanced moments at these three points. The algebraic sum gives the end moments in each span resulting from the original loading. It is a simple matter to sketch in the positive moments and to calculate shears and reactions by the methods of statics.

While familiarity with a method is likely to prejudice one in favor of it, the writer is inclined to grant that the method of Professor Cross is the simpler of the two to apply. However, both are much less cumbersome than

the majority of methods of frame analysis usually encountered in the literature of the subject. It is to be kept in mind that the simplicity of both methods is due in part to the fact that they apply to the type of structure in which there is no displacement of joints or supports.

No originality is claimed by the writer for the idea of distributing the unbalanced moments, but the method of distributing such a moment throughout a frame was given a dozen years ago (1918) in *Bulletin 108*. The writer's method is quite similar to that[4] of H. M. Hadley, Assoc. M. Am. Soc. C. E., except that the procedure given herein for distributing the moment of the unbalanced couples is simpler and more easily followed. The equations given herein will apply to any type of beam or open frame, with various degrees of restraint at the ends of members, as well as for closed frames under conditions of symmetry.

WILLIAM A. OLIVER,[5] ASSOC. M. AM. SOC. C. E. (by letter).[5a]—The writer has read this paper and its discussion with much interest. There is a general human tendency to think that best which is one's own. This is as true of knowledge as of some more concrete possession. The structural analyst becomes familiar through use with one of the several methods of determining the moments in a statically indeterminate structure—slope and deflection, area moments, or what not. Being familiar with that method he concludes from first study that the new one is complicated and unwieldy. In this he is mistaken.

The writer does not intend to intimate that he wishes to detract from the value of the older methods. He received his first understanding of statically indeterminate structures from a thorough course in slope and deflection. However, being somewhat familiar with both methods, he realizes the immense saving of time that results with the use of Professor Cross' method. The short time required for the solution of a continuous beam by moment distribution and successive approximations, as compared with the time needed to solve the same beam by the equation of three moments, is surprising, and the first method may be just as accurate as the second.

Furthermore, the number of seemingly complex problems which may be successfully attacked by the method seem limitless—haunched beams (that is, beams continuous over supports with the moment of inertia varying within the span), continuous arches on elastic piers, and many others.

The writer believes that Professor Cross has discovered something which will add immeasurably to the field of structural analysis and design.

[4] *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 1396.
[5] Champaign, Ill.
[5a] Received by the Secretary, November 26, 1930.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

SOME ASPECTS OF WATER CONSERVATION

Discussion

BY MESSRS. E. C. EATON, AND A. W. SIMONDS

E. C. EATON,[1] M. AM. SOC. C. E. (by letter).[1a]—The writer has read with interest and benefit the excellent paper by Mr. Sutherland. The subject is particularly applicable to many of the problems facing engineers in Los Angeles County, California, where lack of water is the only limitation to expansion and prosperity, and where regulation and conservation by dams of various types are absolute necessities.

California is known generally in the East as an arid or at least a semi-arid State, while the truth is that the State enjoys within itself many unparalleled possibilities for water conservation, the development of which has hardly been scratched on the surface.

Take Los Angeles County as an example; it consists of a broad sloping valley area with a background of high mountains roughly equal to the valley area, from which ages of erosion have built deltas to known depths of more than 1 000 ft. These tremendous cones of gravel, sand, cobbles, and large boulders, forming the underlying strata of practically the entire valley areas, have been built by erosion of the many violent stream flows from more than one hundred and fifty separate water-sheds in the County, each with its individual stream channel debouching on the plains. Many of these water-sheds are small, more than 10% of them being individually less than 3 sq. miles in area, while the largest is only 222 sq. miles in total area. The annual fluctuation of stream flows is great, ranging from 8 to 358% of normal in the thirty-five years for which authentic records are available.

The low average annual rainfall (about 21.1 in.) has been insufficient to wash off completely the heavy cover of disintegrated loose material which, were the rainfall greater, would have already been washed down, lowering the mountain areas and correspondingly building up the valley areas. This loose material of disintegrated granite is held together solely by the root growth of the mountain vegetation, consisting of grasses, chaparral, and some timber.

NOTE.—The paper by R. C. Sutherland, Assoc. M. Am. Soc. C. E., was published in September, 1930, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[1] Chf. Engr., Los Angeles County Flood Control Dist., Los Angeles, Calif.

[1a] Received by the Secretary, November 7, 1930.

During eight months of the year rain is practically unknown and during this period the fire risk is imminent, requiring a most thorough system of policing and fire patrols by Federal and County agencies.

Fires leave the mountain areas without the binders of brush, and even moderate storms following such fires cause a heavy débris flow on to the valleys. A storm having a duration of 10 min. and an intensity of less than 2 in. in 1 hour has been known to bring down as much as 25 000 cu. yd. of débris from a single square mile of water-shed.

The tremendous area and great depth of the gravel cones are in reality of the greatest benefit to this Southern County. These gravel areas, sloping from the base of the foothills to the ocean, are cut by dikes of clay and impervious materials which by damming the underflow to the ocean form a series of underground reservoirs, fed partly by the water issuing from the mountains and partly by the direct valley rainfall. These underground reservoirs are responsible for the entire early development of the County and supply two-thirds of the water used in the County to-day. By the construction of the impermeable covering of the valley, modern development of buildings and roads has not only deprived the underground reservoirs of the natural replenishment, but has increased the rate of run-off.

The keynote to the flood protection and conservation of waters in this County lies in a series of regulating reservoirs in the mountains ironing out the peak flows to a rate of flow which will not be of a destructive nature, but will be of such a rate that the gravels may absorb the stream flow and cause it to pass into the underground basins.

Twelve of these structures have been built to date (1930) and twice this number will ultimately be required ranging from almost every known type of arch dam, including the highest dam in the world (the Pacoima Dam) to earth, rock-fill, gravel-fill, and gravity concrete dams.

The costs of the remaining dam sites in Los Angeles County mountain territory are relatively high and of small capacity when compared to storage sites in many other localities, the cheapest mountain dam built by the Flood Control District costing about $60 per acre-ft., a few being as much as $500 and more, with an average of more than $100. Many "off stream" storages are available, however, ranging from $20 to $50 per acre-ft. of storage space, some of which have been constructed and many more of which are still to be built.

While these costs may appear high the reservoirs are not, in actual practice, used for direct storage. They are merely used for regulation or "ironing out" of peak flows down to a rate such that they can be absorbed by spreading grounds below. These spreading grounds are built over pervious areas overlying the underground reservoirs of which the capacity for water is enormous compared with surface storages. Each of the flood-control regulating dams, therefore, may regulate several times its total capacity in a single year.

In its search for additional regulating reservoirs which are recognized as necessary for the fullest development of the underground reservoirs, the Los Angeles County Flood Control District has investigated scores of dam and reservoir sites, and has felt the need of some satisfactory short-cut method

of comparing the capacities and approximate costs of a number of possible sites. The methods proposed by Mr. Sutherland, therefore, have been studied with great interest and tested to a limited extent as to their application to local conditions.

Mr. Sutherland sets forth a method by which it is possible to compare quickly the merits of various locations and types of dams as regards cost per unit of storage capacity alone. This applies equally well to all uses, whether conservation, regulation for flood control, or power, or any other use for which water is stored. The amount by which the results attained by this approximate method differs from those by the more exact and slower methods depends upon the variation from a straight line of the capacity curve platted on logarithmic paper and upon the judgment used in establishing a dam site cross-section which is equivalent to the true cross-section. The method appears to be quite satisfactory in the case of large reservoirs.

In the case of relatively small reservoirs, requiring high dams, such as those built by the Los Angeles County Flood Control District, the errors introduced by the approximations are likely to be relatively large. Mr. Sutherland cites three uses for the alignment chart, Fig. 4. Puddingstone Reservoir, which is the only County Flood Control Reservoir constructed to date that falls within the limits of the chart, was studied from the standpoint of Use No. 1. Adopting the value of m determined by a depth of water of 132 ft. at the deepest point, the capacities corresponding to various depths as shown by the alignment chart and by the ordinary capacity curve are as given in Table 8 for comparison.

TABLE 8.—COMPARISON OF RESERVOIR CAPACITIES

Maximum depth of water, in feet	From alignment chart, Fig. 4, in acre-feet	From capacity curve, in acre-feet	Variation, percentage
57	920	1 056	12.9
82	3 493	3 639	4.0
107	9 267	9 406	1.5
132	20 012	20 012	0

The variation results from the fact that for any of the reservoirs of the District, the value of m varies when different depths of water are used in its determination.

Many considerations determine the justified expenditure for a dam. The benefits to be derived from, and the cost of, the project as a whole are the determining factors. The cost of channels below flood-control dams depends on the degree of control effected by the reservoir. The justified total cost for the system depends on the amount of damage prevented through its operations. It is estimated that without protection, $50 000 000 of direct and indirect damage would result from a flood on the San Gabriel River equal to that which occurred in 1914. It is obvious that a reservoir on this river would be in reality of greater value than a reservoir on some other stream on which little or no damage would result, but which was classified as excellent.

From the conservation standpoint the cost per unit of water delivered to the consumer is the determining factor. Systems designed for conservation may be beneficial in two ways:

1.—By causing the retention for use of water which would otherwise be wasted; and,

2.—By effecting a decrease in cost of delivery to consumers through holding the water at more favorable locations until such time as it is to be used.

The expense justified in carrying out the first of these items depends upon the use to which the water is to be put and the value of crops which it is possible to raise under the existing climatic and soil conditions. In general, it might be stated that the least cost at which water can be supplied in sufficient quantity to the entire area under consideration is the factor which fixes the cost justified in any case. Thus, in the case of domestic supply, a reservoir costing much more per acre-foot stored might be more necessary and, therefore, more valuable than a cheaper reservoir supplying some area where reservoir sites are plentiful and water easy to obtain.

In the area covered by the Los Angeles County Flood Control District, the storage capacity available is limited, not only by the general steepness of the topography, but by the fact that foundation conditions preclude the use of many dam sites. This limited storage must be used for two purposes, flood control and conservation. In many cases the total storage space available is insufficient to control the stream completely. The storage required for economical control is of greater importance than the cost of this storage per acre-foot.

The foregoing shows that reservoir sites can not be classified as excellent, good, average, or bad, depending upon their cost per acre-foot of capacity. However, where other conditions are equal, two or more dam sites can be compared and one can be shown to be better in this respect than the others. Mr. Sutherland states that these other factors must be considered, but their importance far outweighs that of cost of storage per acre-foot, in the case of flood-control dams in an area of high productivity, such as this one.

Nevertheless, a comparison of capacities and costs of possible sites, between two or more dam sites has often been found necessary here, even though the unit capacity costs of such sites are so high as to eliminate their consideration in other parts of the United States where conditions are more favorable.

In general, to make such a comparison, it is desirable to answer either or both of two questions, other things being equal:

(*a*) Of two or more dam sites the reservoirs of which overlap and which serve about the same water-shed, which will be most economical?

(*b*) Where two or more small reservoirs compare with one larger reservoir for the same water-shed, what combination will be the most economical?

The following short-cut method has been developed by the Los Angeles County Flood Control District, largely through the efforts of C. B. Gibbons, Designing Engineer, and has proved very satisfactory for preliminary studies, serving to eliminate from further field investigation all the less advantageous sites. The method may be described in fifteen successive steps as follows:

1.—A topographic map is prepared. Aerial surveys give results close enough for such investigation, and have been used extensively.

2.—The topographic map is divided into zones along the stream bed by vertical planes (sometimes broken in plan), cutting along every important ridge and roughly segregating portions of the reservoirs between successive bottle-necks or narrows. (See Fig. 19.)

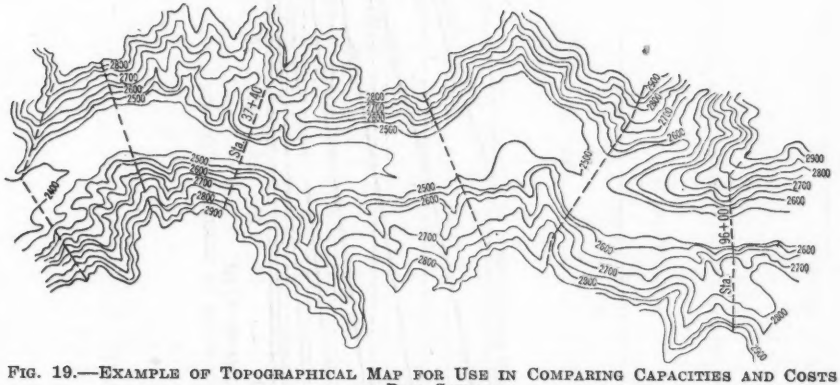

FIG. 19.—EXAMPLE OF TOPOGRAPHICAL MAP FOR USE IN COMPARING CAPACITIES AND COSTS OF DAM SITES.

3.—For a distance of 300 to 400 ft. vertically above the stream bed, each contour, by zones, is measured by a planimeter.

4.—Beginning at the point at which each contour crosses the stream bed, the partial areas within this contour are cumulatively totaled for each zone, so that at each bottle-neck or narrows, the relation between depth of water and reservoir area is determined.

5.—Beginning at the stream bed at each bottle-neck or narrows, the cumulative areas are cumulatively totaled, and these totals, multiplied by the contour interval, give the usual data for a reservoir capacity curve at each such point.

6.—To serve as abscissas in Fig. 20 (which is described in Step 8), distances along the stream bed are roughly measured, and both bottle-necks and contours noted:

7.—Using stream-bed distances as abscissas and acre-feet of reservoir capacity as ordinates, graphs are drawn (to 300 or 400 ft. above stream bed) for each contour measured. (See solid lines in Fig. 20.)

8.—From the reservoir capacity curves thus drawn and the points at which each contour crosses the stream bed, other graphs are drawn for each height of dam (in even multiples of the contour interval). (See broken-line curves in Fig. 20.)

9.—Graphs such as Fig. 20 portray continuously, over the range being investigated, the relation between reservoir capacity and height of dam. This graph has been called a capacity profile. By inspection, certain narrows or bottle-necks may be selected, which are favorable because the capacity is large as compared with the cross-section.

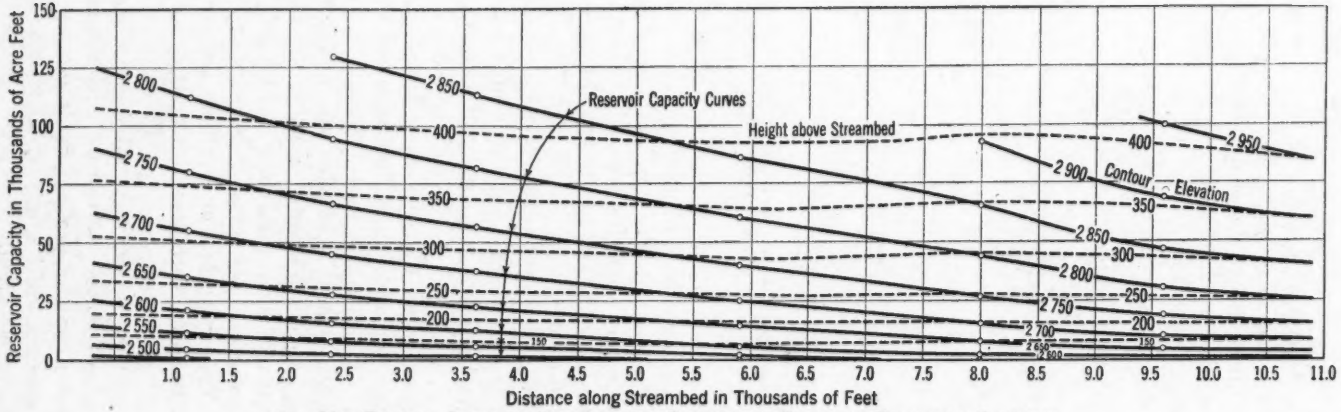

FIG. 20.—TYPICAL CAPACITY PROFILES FOR CONTINUOUS STUDY OF RESERVOIR CAPACITY.

Thousands of Cubic Yards for Earth and Rock Fill Dams.
Hundreds of Cubic Yards for Concrete Gravity Dams.

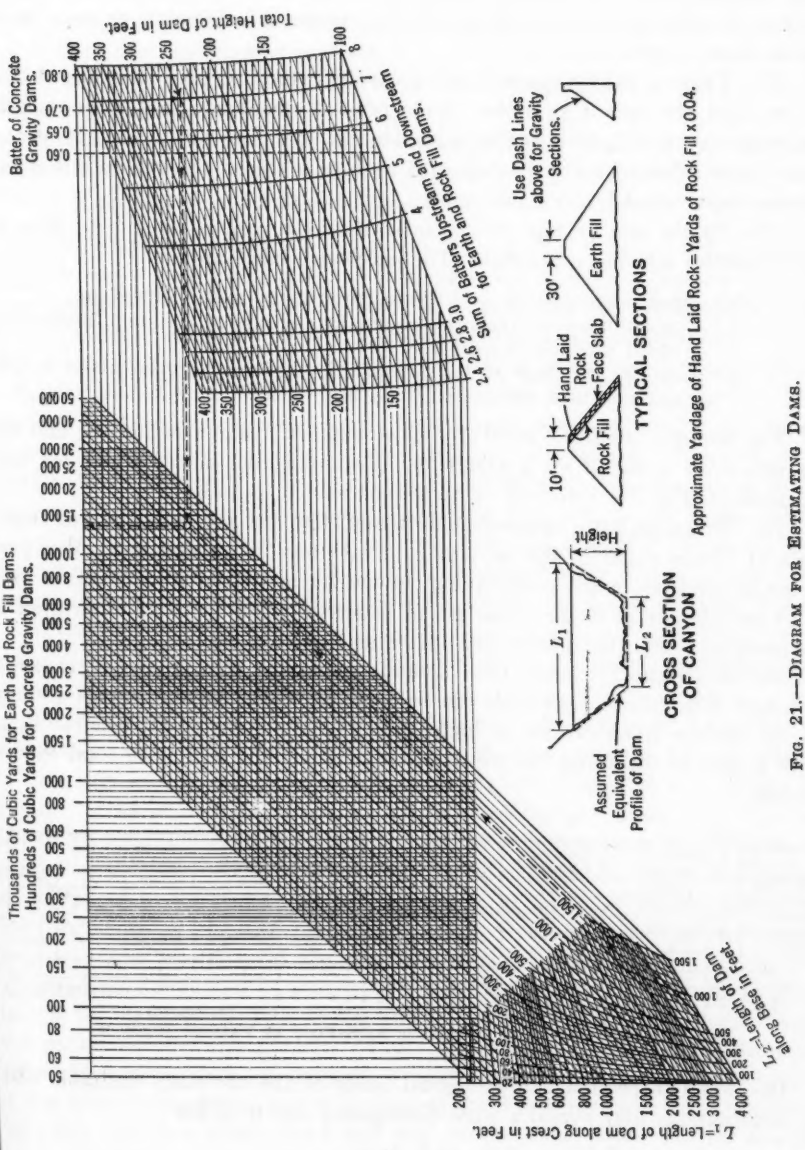
Total Height of Dam in Feet.
Batter of Concrete Gravity Dams.
Sum of Batters Upstream and Downstream for Earth and Rock Fill Dams.
Thousands of Cubic Yards for Earth and Rock Fill Dams.
Hundreds of Cubic Yards for Concrete Gravity Dams.
L_1=Length of Dam along Crest in Feet.
L_2=Length of Dam along Base in Feet.
Use Dash Lines above for Gravity Sections.
Hand Laid Rock Face Slab
Rock Fill
Earth Fill
TYPICAL SECTIONS
Height
CROSS SECTION OF CANYON
Assumed Equivalent Profile of Dam
Approximate Yardage of Hand Laid Rock=Yards of Rock Fill x 0.04.

FIG. 21.—DIAGRAM FOR ESTIMATING DAMS.

10.—At such selected points, a cross-section of the canyon is drawn, and for different heights of dam, a trapezoid is constructed, which roughly averages the cross-section. Increasing weight must be given to the lower portions where the dam is thicker. A reasonable amount of stripping or excavation must be assumed.

11.—Prior to field inspection and geological study, it may be assumed that these sites are equally feasible. Some type of structures, such as a rock-fill or concrete gravity dam, is then estimated at each site to various elevations, care being taken to allow for increased average width at the stream bed where higher dams straddle a rounded nose.

12.—By the use of Fig. 21 the quantities for various heights of dam at each feasible site are computed. The successive steps are:

(*a*) Draw a cross-section of the canyon on the axis of the dam.
(*b*) Construct an equivalent straight-line profile of the dam, neglecting small deviations near the top.
(*c*) Find the yardage at the top of the diagram for the given height and assumed values of L_1, L_2, and batter.

For example, let $L_1 = 2\,000$ ft.; $L_2 = 1\,400$ ft.; height = 250 ft.; and the batters 4 on 1 and 3 on 1 (total 7). Following the dotted lines, for this example in Fig. 21, a volume of 13 400 000 cu. yd. is indicated.

13.—By assigning a reasonable unit cost, the cost per acre-foot (at intervals of height equal to the contour interval) may be determined. For purposes of comparison the costs in Fig. 22 are estimated for gravity dams with a 10 on 8 batter at $8 gross per yd. of concrete. The depth of excavation is assumed as 50 ft. Quantities are estimated from Fig. 21.

14.—From the foregoing data assuming that the total reservoir capacity has been determined, a schedule can be prepared which makes possible a study of the various combinations of high dams and low dams at all feasible sites, with a view to obtaining the requisite capacity for the minimum total expenditure.

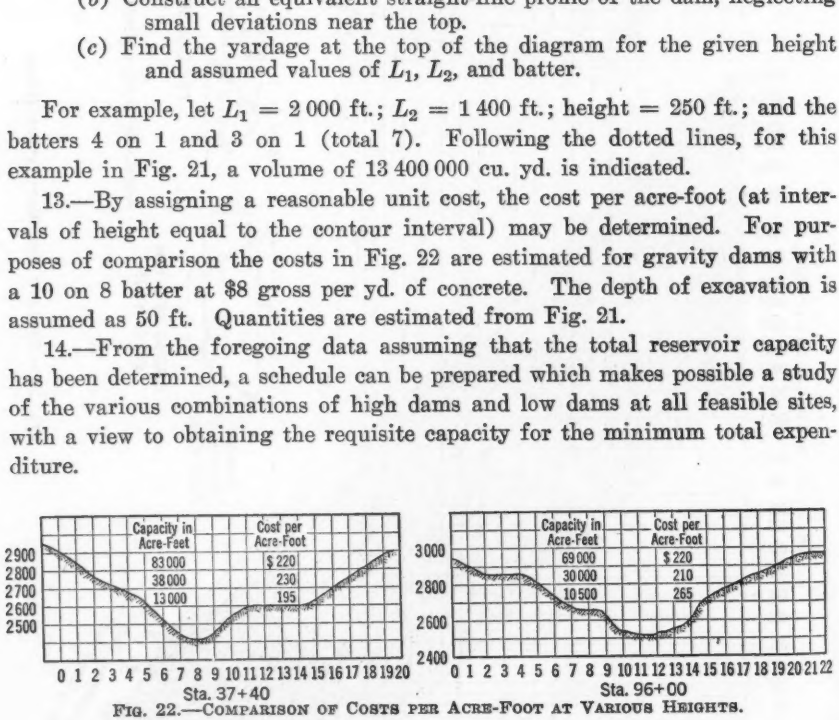

FIG. 22.—COMPARISON OF COSTS PER ACRE-FOOT AT VARIOUS HEIGHTS.

15.—Field inspection and geological study of the obviously desirable sites or combinations may reveal a wide discrepancy due to either:

(*a*) The availability of an arch dam;
(*b*) A considerable difference in the requisite excavation or stripping; or,
(*c*) The existence of faults which exclude all except a more expensive flexible dam.

In the latter case a revision of the comparative cost figures at the different sites may be made, retaining only the data which correlate capacity with height of dam.

The foregoing method has been described as an alternative means of arriving at the same objective as that sought by Mr. Sutherland's methods, and is felt to be more applicable perhaps to the special conditions existing in Los Angeles County as outlined.

Mr. Sutherland's paper has been the object of much interest in Los Angeles County and will undoubtedly prove of great value, particularly where the method is applied to reservoirs of considerable size and uniform cross-section.

A. W. SIMONDS,[2] ASSOC. M. AM. SOC. C. E. (by letter).[2a]—This paper briefly covers a great many points of interest in the problem of water conservation, indicating that the author has spent much time in the study of this subject. Water conservation is a geologic as well as an engineering problem. Too much emphasis can not be placed on this phase, particularly where a failure would involve the loss of considerable life and property. The engineer who disregards the geologic factors may invite disaster to his work. Some glaring examples of this are the failures of the Colorado River Dam at Austin, Tex., and the St. Francis Dam in California. The Colorado River Dam was built across a badly shattered fault zone which was inadequately excavated and not grouted. The result was that a section of the dam over the fault zone slid out. Some years later the dam was repaired, but as a storage reservoir the project was a failure, owing to the rapid loss of capacity due to silt deposition. If the services of a competent geologist or an engineer with a knowledge of geology had been engaged before the dam was built, these results might have been foreseen.

Successful projects in one locality might be failures elsewhere, due to different geologic conditions. Such dams as the Cheesman and Barker Dams in Colorado and the Gibson Dam in Montana are successful as a means of conserving water in their respective localities. They are situated high in the Rocky Mountains where their reservoirs are fed with clear water for the greater portion of the year. If these dams were placed across some heavy silt-bearing streams, such as the Brazos or Colorado Rivers of Texas, or the Rio Puerco of New Mexico, their reservoirs would lose capacity rapidly due to silt and would become useless as agents of water conservation.

The dam site should be thoroughly explored by diamond drilling, particularly if the structure is to be of masonry or concrete. Even then an engineer's eyes cannot see very far into rock. In the case of the Gibson Dam, in Montana, sub-surface exploration was made by drilling. In excavating for the structure, it was found unnecessary to cut as deep at the crown section as was originally intended, but at a point about half way between the crown section and the north abutment, it was necessary to cut deeper than the original line of assumed excavation.

[2] Assoc. Engr., U. S. Bureau of Reclamation, Denver, Colo.

[2a] Received by the Secretary, December 15, 1930.

The curves for submerged area (Fig. 6), site area (Figs. 7 and 8), and volume (Figs. 10 and 13), are interesting, but are of doubtful practical value. Dams and reservoirs are expensive structures, and their cost warrants a careful engineering investigation. A topographic survey of the reservoir and dam site is one of the first requirements. A carefully made topographical map will be found much more reliable for computing volumes in dams than the use of curves. This is particularly true in the case of the Deadwood Dam in Idaho. If the location of this dam were to be shifted a short distance either up stream or down stream its volume would be materially changed.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS AND DISCUSSIONS

STRESSES IN GRAVITY DAMS BY PRINCIPLE OF LEAST WORK

Discussion

BY MESSRS. C. A. P. TURNER, DONALD P. BARNES, AND ANTHONY HOADLEY.

C. A. P. TURNER,[1] M. AM. SOC. C. E. (by letter).[1a]—Mr. Jakobsen's effort to analyze the stresses in gravity dams seems predicated upon fundamental misconceptions of the ordinary theory of flexure. For example, he refers to the ordinary theory of flexure as "classical". The customary lack of precision in stating the assumptions on which this theory is based combined with a total disregard of elementary principles by textbook writers in its elucidation, seems to predicate an antiquity of thousands of years, although its origin in fact dates back less than two and one-half centuries.

The earliest effort to investigate the resistance of beams is found in Galileo's "Discoursi e Dimostrazioni Mathematiche" published in 1638. Erroneously Galileo assumed stresses of one kind only, with an axis of rotation at the bottom of the cantilever beam considered. In 1686, Mariotte published results of tests showing that Galileo's theory was not in accord with experiment. He noted that while part of the fibers are stretched, others are compressed, and assumed without proof ("on peut concevoir") that one-half the fibers are stretched and one-half are compressed. Discussing the tests of Mariotte, Leibnitz concluded that the resistance of the fibers was in proportion to their extension.

The further development of this theory embodying the hypothesis that the extension of the fibers is in proportion to the bending moment and to their distance from the center of gravity of the section as well is the Bernoulli-Eulerian theory or the common but inaccurate theory of flexure taught in engineering schools to-day. The assumptions involved for a rectangular section are:

1.—As many fibers are stretched as compressed.
2.—The neutral plane is the gravity axis of the section.
3.—A plane section normal to the gravity axis before bending remains normal to this axis after bending.

NOTE.—The paper by B. F. Jakobsen, M. Am. Soc. C. E., was published in September, 1930, *Proceedings*. Discussion of the paper has appeared in *Proceedings*, as follows: September, 1930, William Cain, M. Am. Soc. C. E.; November, 1930, by Messrs. William Gore, L. J. Mensch, Johannes Skytte, and Lars R. Jorgensen.

[1] Cons. Engr., Minneapolis, Minn.

[1a] Received by the Secretary, October 29, 1930.

4.—A plane section normal to the gravity axis before bending remains a plane section after bending.

5.—The material of the beam obeys Hooke's law that strain is proportional to stress.

Internal horizontal shearing stress is correctly treated as the reaction to the summation of the horizontal fiber stresses. The center of action of the fiber stresses is 0.7071 times the half depth above the gravity axis and that of the half parabolic shear stress area is approximately 0.348 times the half depth above the same axis. Hence, the summation of a triangular fiber stress area results in a parabolic shear stress area and the neutral plane must be lowered 0.179 times the half depth below the gravity axis at the end of the beam, so that the reaction of the shearing stresses may be opposite the action of the fiber stresscs, as required by Newton's third law.

Therefore, there are, in fact, more fibers compressed than are stretched in a simple beam, and the linear law of fiber stress intensity in the tension zone differs from that in the compression zone except at sections of zero internal shear strain. Since the positive shear in the compression zone at the neutral plane must equal the negative shear in the tension zone, the rate of increase of the fiber stress must be greater in the tension zone than in the compression zone except at that normal section at which the true neutral plane is tangent to the gravity axis of the beam, namely, the center of a symmetrically loaded simple beam and at the support of a cantilever.

Weisbach in his "Mechanics of Engineering", appears to have been the first to note that equilibrium of the shearing stresses must be taken into account in the problem of determining the position of the neutral plane, but he failed to solve the problem.

Assumptions (3) and (4) of the ordinary theory are inconsistent with and contradictory of Assumption (5). They present the grotesque presumption that the shearing modulus may be treated as infinitely great in lieu of its normal finite value of $\frac{3}{8}E$ for homogeneous materials.

Assumption (4) is largely innocuous; thus a normal plane before bending remains substantially plane after bending in the web of a wide flanged I-beam, yet the anti-clockwise rotation of the plane from the normal renders the error in computing the elastic deflection by the ordinary theory equal to from 15% for ordinary spans to 50%, dependent upon the relative length to depth of the beam considered.[2] By disregarding shear strain the deflections of simple beams are computed as less, and those of continuous beams as greater, than actually occurs.

For the development of the exact theory of flexure it is necessary only to assume that the material obeys Hooke's law. The linear law of fiber stress distribution upon the normal central cross-section of a symmetrically loaded simple beam and a like section at the supports of a cantilever follows inevitably from this hypothesis.

For equilibrium under internal shear the sum of the compressive fiber stresses parallel to the neutral plane must equal the sum of the tensile stresses

[2] These ratios apply to steel I-beams with broad flanges.

parallel to the same plane acting upon any section normal to the neutral plane along x. Because these positive and negative forces form a couple, the moment of which is the same about any point in its plane of action, the hypothesis of a neutral plane at the gravity axis is unnecessary in developing exact summation equations of flexure.

Let the tensions and compressions parallel to the neutral plane be defined as moment stresses. Then, for a simply supported uniformly loaded beam, if w equals the uniform load per unit of length; S, the external shear force at any section; M, the applied moment; a', the slope; and D, the deflection caused by moment stresses: $S = \Sigma\, w$; $M = \Sigma\, S$; $a' = \sum \frac{M}{E\, I}$; and $D = \Sigma\, a$. The moments, slopes, and deflections of the ordinary theory are then determined at once by simple arithmetic, noting the geometrical relation that a parabolic moment area is two-thirds of its rectangle and that the third degree slope area is five-eighths of its rectangle in the computation.

It appears therefore that the ordinary theory of flexure is an exact analysis of the slopes and deflections accruing from moment stresses, that is, those parallel to the true neutral plane. Such stresses in a horizontal beam cause a clockwise rotation with respect to the vertical of every plane section normal to the true neutral axis when looking toward the left side of the simply supported beam. Shear strain, on the contrary, gives rise to anti-clockwise rotation tending to relieve the moment stresses; hence increased deflection follows until static equilibrium is restored at mid-span. Therefore, a, the true slope, is greater than $\sum \frac{M}{E\, I}$ in the simple beam or cantilever.

Because shear strain at the neutral plane is the reaction to the moment stresses and because the anti-clockwise rotation results therefrom and is limited by the relative cross-section and elastic properties of the material, the following relations may be written: True slope, a, equals moment slope, $a' \times 1 + g$; and true deflection, Δ, equals moment deflection, $D \times (1 + g)$, which constitute the only modification that need be made in the ordinary summation equations of beam theory to reduce them to values in accord with exact analysis.

The factor, g,[3] is a fraction that is positive for simple and cantilever beams, but negative for continuous and restrained beams. It is a constant for a given beam regardless of kind or character of loading, but varies for different beams dependent upon relative length and cross-sections of flange and web.

Shearing strains, by exact analysis, differ from those determined under the arbitrary assumptions of the common theory far more widely than the moment stresses. In the common theory of flexure the assumption that internal shear strain may be disregarded is false, because shear failure occurs in short beams and must be accounted for somehow. Accepting this assumption, textbook writers and building code authors again resort to the erroneous procedure of disregarding Newton's third law.

[3] The computation of g is given in Section II, "Elasticity," pp. 16 to 21.

Although recognizing that horizontal shear strains furnish the necessary stress reactions to the positive and negative moment stresses, in order to permit their joint action as a couple, they assume, erroneously, that internal shearing stress is the reaction to external shear force because rotational strain (pure shear) presents the same rotational resistance through 360°, as shown by Cauchy; that is, horizontal shear = vertical shear at the neutral plane.

Because the horizontal shears are reactions equal to the summation of the moment stresses they cannot be equal also to the vertical external shear force since the moments under the accepted equations are equal to the summation of the shears and, therefore, cannot in general equal the external shear force at a section. To state the criticism with Euclidian finality, "two things which are not equal to the same thing cannot be equal to each other".

A graphic diagram of an unsymmetrically loaded beam best illustrates this divergence. With a concentrated load at P, in Fig. 17, distant $0.3\,L$ from the left support, the external shear passes through zero and changes sign under the load, P, and there the moment is a maximum; but at AA, a plane which divides the moment area in equal parts, internal shear strain must pass through zero where the bending moment is not a maximum and the external shear force is not zero.

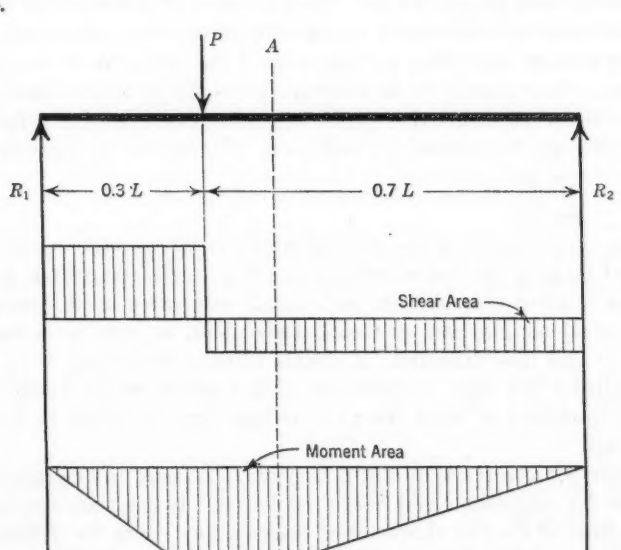

FIG. 17.

Because the moment area to the left of AA equals the moment area to the right, the summation of bending to the right and left of AA must be equal as action and reaction and the shear stress parallel the neutral plane is zero at AA and sums to the right and left toward the end of the beams. Likewise

in a horizontal cantilever the summation of the horizontal shear stresses commences from zero at the support, increasing toward the free end in a manner analogous to the summation from the mid-span toward the end of the simply supported beam.

In a cantilever, horizontal shear strain passes through zero at the neutral surface at the support and the external shear force is a maximum at this point. On the contrary, in a simply supported beam the internal shear strain at the support is a maximum and the external shear force is also a maximum. Therefore, the simply supported beam is capable of resisting much smaller unit shear forces per unit of cross-section area than the cantilever, the continuous, or the restrained beam of the same cross-section. In the case of the cantilever or a continuous beam, where the internal shear at the neutral axis is zero, the external shear force must be resisted by the components of the principal stresses and the diagonal tensions and compressions due to the sharpness of curvature.

Mr. Jakobsen's painstaking investigation proceeds on the common but erroneous practice of confusing external shear force with internal shear strain. Unfortunately, therefore, it is founded on the theoretical "quicksand" of the differential calculus of assumptions and the integral calculus of presumptions rather than on the bed-rock of fundamental principles and the sound logic of the "Principia".

The linear law of fiber stress intensity at a section normal to the neutral plane is an inescapable consequence of the assumption of homogeneous elastic material subject to Hooke's law. Any supposed proof to the contrary indicates error. In the case of Mr. Jakobsen the error is embodied in the misapplication of both the common theory and exact theory of flexure to the triangular dam. The linear law applies to a section normal to the neutral plane which, at the base, coincides with the gravity center of the mass. By erroneously assuming a horizontal plane diagonal to the gravity axis the designer concludes that fiber stresses of flexure are much less than their maximum intensity; and the linear law of fiber stress intensity does not apply, and is not presumed to apply, to a section diagonal to the normal under the ordinary theory.

From the preceding discussion it appears that whether the exact theory of flexure or the common approximate theory is applied, the maximum bending stresses in the triangular dam will be determined to be of like magnitude.

The first step is to locate the critical section. The gravity axis of the triangular mass makes an angle, $\frac{1}{2}\phi$, with the vertical. The critical section inclines upward from the bed at the base of the dam at an angle of $\frac{1}{2}\phi$ to the horizontal. Obviously, this resisting section is smaller in area than the horizontal section considered by Mr. Jakobsen; the lever arm of the dead weight is less and the maximum stresses will be greater than those erroneously computed on the assumption that the critical section is horizontal and diagonal to the gravity axis. The variation in stress must follow the linear law on the critical section if the material is elastic and obeys Hooke's law. Its rate of variation on the diagonal section is an academic matter without practical interest to the busy constructor.

In attempting to point out some of the misconceptions and erroneous assumptions presented in textbook theory, it will be well to call attention to the general misunderstanding of the exact nature of Poisson's ratio.

The lateral contraction of elastic material under longitudinal strain is brought about by the inclination of the planes of greatest shear to the axial direction of the stress. This inclination is a constant at 45° in all cases of homogeneous strain, but it differs from 45° in all cases of heterogeneous strain. Therefore, $\frac{1}{m}$ for a case of heterogeneous strain is different and distinct from $\frac{1}{m}$ in the case of homogeneous strain, a fundamental relation which is lost sight of in Mr. Jakobsen's discussion.

The determinate structure is one in which the reactions are determined by the law of gravitation and the principle of the simple lever; hence the deflection and work of deformation is the greatest that the resistance of the material permits under the principle of the lever.

The indeterminate structure, on the contrary, is governed by the principle of rigidities, which asserts that where there are different paths by which the load may travel to the support, it divides itself between these several paths strictly in proportion to the respective rigidities, and when so divided the work of deformation will be less than with any other or different division. Thus arises the so-called "law of least work."

The old preparatory school algebras taught that if a function of the two variables is plotted when the value of Y along X is a maximum, the tangent to the curve becomes parallel to the axis of abscissas; that is, its angle thereto equals zero and its tangent, $\frac{dy}{dx}$, is also zero. In like manner where the value of Y along X is a minimum the tangent to the curve is likewise parallel to the axis of abscissas and its angle therewith passes through zero.

These relations comprise the old-fashioned criterion for maximum and minimum values. Men of the modernistic school and textbook writers insist upon treating this old and familiar principle in the theory of work as the derivative of the work function and render it so mysterious that maximum values are confused with minimum, and the precise mathematical distinction between the determinate and indeterminate structure is lost sight of, as appears in Mr. Jakobsen's discussion.

Thus, at the outset, Mr. Jakobsen correctly classifies the section of the gravity dam as determinate under the law of gravitation and principle of the lever. Then he proceeds to derive equations of resistance upon this basis and, finally, assumes that they may be inserted in his work equations and solved by the "principle of least work", thus confusing the diverse characteristics of the determinate and indeterminate structure.

If gravity dams are to be built, the maximum stresses at the critical section should be computed with a reasonable or practical degree of accuracy. This cannot be done if sections other than the critical are selected. Taking a horizontal section of the dam, when the normal to the neutral plane makes an angle thereto, results in a moment of inertia far greater than at the critical

section; so that there is a large error on the unsafe side in the computation so made; and this, according to Mr. Jakobsen, is the ordinary but inaccurate procedure of hydraulic engineers in general.

The writer's opinion of the gravity dam is that it may be regarded as reasonably economical where the head of water to be impounded does not much exceed 15 or 20 ft. For a higher dam the gravity type which has been handed down from an ancient period is highly uneconomical. It lacks the fundamental characteristics that are desirable from the standpoint of safety. Thus, with the gravity type, as the head increases the over-turning resistance decreases, but with a properly designed reinforced concrete dam having an inclined up-stream face, on the contrary, as the pressure increases the over-turning resistance by that pressure increases in like ratio, and the rational elements of stability are combined with greater economy. Why hold on to that which is obsolete instead of that which is up to date and modern? This is a question to which hydraulic engineers should give consideration in the light of the wastefulness and reduced safety in their customary manner of building.

DONALD P. BARNES,[4] JUN. AM. SOC. C. E. (by letter).[4a]—The accumulated evidence that the assumption of linear stress distribution is in error may be regarded as conclusive. Mr. Jakobsen's analysis is theoretical proof for the case of a dam on a rigid foundation, and the various experimental results cited by him are factual evidence of the inaccuracy of the old theory. That this inaccuracy is appreciable, and that the real maximum stresses are materially greater than indicated by the straight-line theory are likewise justifiable inferences. There remains, however, considerable uncertainty as to the closeness with which the results obtained by Mr. Jakobsen's method approximate the true stresses.

This uncertainty arises, perhaps primarily from the assumption of a rigid foundation. The author has himself observed in the closing paragraph of Appendix III that no account is taken of irregularities of the base, and also that "* * * as Dr. Vogt has shown, the foundation does not remain plane when the stresses are linear."[5] The philosophy of the principle of least work is probably entirely acceptable, but the statement of its application should be the condition that the sum of the internal work of the dam and of the foundation shall be a minimum. It does not follow that the work of either, alone, would be minimum.

The inclusion of the internal work of the foundation may be an insurmountable mathematical obstacle. Its neglect, however, would seem to render the utility of Mr. Jakobsen's results largely dependent upon experimental confirmation. In this connection, the results of a physical investigation recently completed on a celluloid model may be of interest.[6]

The model was cut in the proportions shown in Fig. 18, the dam portion being exactly the same shape as the first profile analyzed by Mr. Jakobsen,

[4] Designer, Pasadena Water Dept., Pasadena, Calif.

[4a] Received by the Secretary, November 12, 1930.

[5] *Proceedings*, Am. Soc. C. E., September, 1930, Papers and Discussions, p. 1637.

[6] "On Stresses in Gravity Dams," Thesis presented at California Inst. of Technology, June, 1930.

but cut in one piece with the base, in order to include the effect of the elasticity of the foundation as nearly as practicable. Loads graduated to simulate water pressure were applied to the the up-stream face. The weight of the structure itself (gravity load) was neglected, as was also the vertical water pressure on that part of the foundation under the reservoir. The condition was that discussed by Mr. Jakobsen in Section VIII.

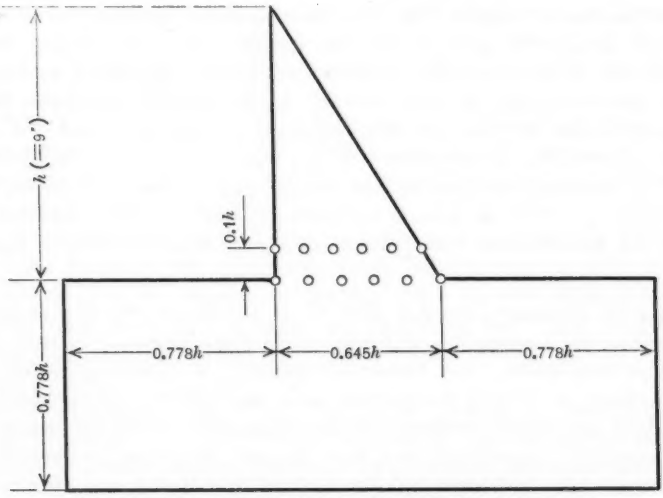

FIG. 18.—SHAPE OF MODEL AND POSITION OF POINTS THE DISPLACEMENTS OF WHICH WERE MEASURED.

The vertical components of the displacements of the points shown were measured with micrometer microscopes. These distances were plotted to an exaggerated vertical scale and curves drawn through the points. The curve of the upper row of points was extrapolated at the down-stream end. Differences between ordinates to points lying in the same vertical line on the two curves, divided by the interval between the rows, were assumed to equal the unit vertical deformations, δ, of corresponding points on the intermediate section (one-twentieth of the height from the base).

From the application of Hooke's law, considering Poisson's ratio,

$$\delta = \frac{n_y}{E} \pm \frac{n_x}{E}\frac{1}{m} \qquad (46)$$

in which, n_y, n_x, E, and $\frac{1}{m}$ are the vertical and horizontal unit stresses, the modulus of elasticity, and Poisson's ratio, respectively, for celluloid. From Equation (46) and the measured values of δ, assuming $E = 295\,000$ lb. per sq. in., $\frac{1}{m} = 0.32$, and $n_x = 58$ lb. per sq. in. (approximate average, assumed constant across the base), a stress curve similar in shape to that of Fig. 19 was found for which the equilibrium equations, $\Sigma n_y = 0$, and $\Sigma M = 0$, constituting a check, were completely satisfied. However, George E. Beggs, M. Am.

Soc. C. E., gives[7] $E = 263\ 750$ lb. per sq. in., and $m = 2.38$ $\left(\frac{1}{m} = 0.42\right)$ for celluloid. This circumstance, combined with the fact that no check of these values was made for the model in question, detracts somewhat from the value of the check noted.

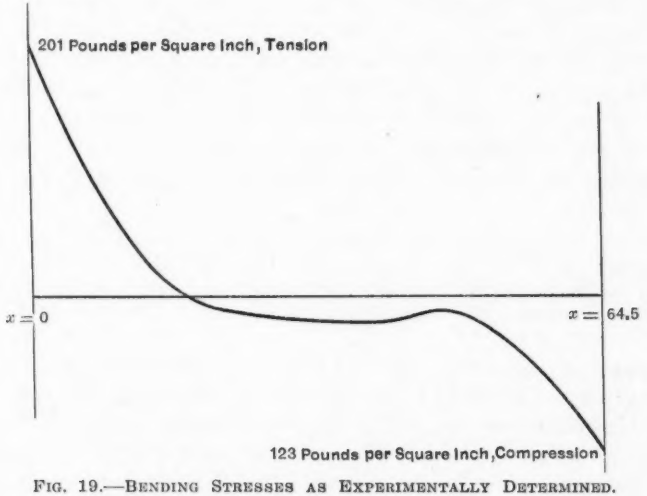

FIG. 19.—BENDING STRESSES AS EXPERIMENTALLY DETERMINED.

The stresses in the hypothetical concrete dam were obtained from the stresses in the celluloid by applying the constant ratio, 0.651, obtained from the relations of the dimensions and loads of the dam and model. Fig. 19 is the curve of these stresses.

For this case of horizontal loads only, it will be seen that the experiment referred to, indicates a maximum tensile stress about twice as great as that found by the straight-line theory and a compressive stress about 25% greater. Mr. Jakobsen's analysis gives a tensile stress about 43% greater and a compressive stress about 20% less than those found by the linear theory.

It does not appear that the experimental confirmation of Mr. Jakobsen's method at present available is sufficient to warrant its use in practical design, particularly inasmuch as its fundamental assumption, that of the principle of least work, is not so applied as to include the effect of the foundation. His paper is, however, a strong argument for the use of much greater conservatism in the selection of a profile than is the present practice.

ANTHONY HOADLEY,[8] ASSOC. M. AM. SOC. C. E. (by letter).[8a]—The author has thrown considerable doubt on the validity of the "middle-third" rule which has been widely used in the design of dams. The results of his theoretical investigations form another piece of evidence, in addition to several others which have been brought sharply to the engineer's attention in the last

[7] Report on Arch Dam Investigation, *Proceedings*, Am. Soc. C. E., Part 3, May, 1928, p. 219.

[8] Asst. Prof., Civ. Eng., Union Coll., Schenectady, N. Y.

[8a] Received by the Secretary December 1, 1930.

few years, calling for a conservative viewpoint in the design of gravity dams. In order to obtain a workable set of equations it has been necessary to set up a formidable array of restrictions. These restrictions are analogous to those used in developing the theory of stress distribution in reinforced concrete. Although the actual stress distribution in either dam or reinforced concrete member will differ from the theoretical distribution, it seems to the writer that the theory is still a useful tool to the designer.

The theory developed by Mr. Jakobsen indicates that many existing dams must be in tension at the up-stream face. That this is possible has been indicated[9] by Searcy B. Slack, M. Am. Soc. C. E., in reporting strain measurements which indicate stresses as high as 250 lb. per sq. in. in tension. In commenting on the Yadkin River Bridge tests, Clyde T. Morris, M. Am. Soc. C. E., states,[10]

"2.—The calculated compressive stress at any section of the rib checks the observed stress more closely when it is assumed that the concrete takes tension, than when it is assumed that the concrete is not active in resisting tensile stresses, even when high tensile stresses have caused cracks in the concrete."

How much stress a horizontal construction joint (which destroys the homogeneity of the mass, and, consequently, limits the applicability of Mr. Jakobsen's equations) will carry, is another question.

The use of a factor of safety of one in the design of dams seems to the writer to be unsafe. He is glad to see a theory presented, which calls for increasing the base width. In 1928, the writer was doing some experimental work in which a silk line (50-lb. test) was used to carry the load. The consequences of a break in the line were not severe. It meant merely an half hour spent in putting in a new piece of line plus the possibility of a cut or bruised hand; but even at that, a very uncomfortable feeling was produced each time the load approached the 50-lb. mark, which would give a "factor of safety" of one. Surely the designer of a dam should protect himself from that "uncomfortable feeling"—or worse—by using a factor of safety materially greater than one. It may be argued that the loads acting on a dam are static and definitely known, but this does not seem to be borne out by evidence produced in two papers recently published.[11]

It is very interesting to notice the slight change in stress-distribution (Table 9) caused by assuming widely different values for Poisson's ratio. The writer wonders what the effect would be if Poisson's ratio varied from one section of the dam to another.

The writer found that, according to the theory developed by Mr. Jakobsen, the width of the base of a dam of triangular section should be 0.82 times the height if the tension is to be avoided at the heel; or, expressed in another way, the resultant force should not fall on the down-stream side of the middle thirteenth of the base. This section closely approximates that suggested by

[9] "Behavior of an Arch During Construction", *Proceedings*, Am. Soc. C. E., November, 1929, Papers and Discussions, pp. 2291–2293.

[10] *Proceedings*, Am. Soc. E. March, 1929, Papers and Discussions, p. 611.

[11] "Hydrostatic Uplift in Pervious Soils", by H. de B. Parsons, M. Am. Soc. C. E., "*Transactions*", Am. Soc. C. E., Vol. 93 (1929), p. 1317; and "Upward Pressure Under Dams: Experiments by the United States Bureau of Reclamation", by Julian Hinds, M. Am. Soc. C. E., *Loc. cit.*, p. 1527.

Maurice Levy. Equations (16), (17), (25), and (26) were solved for the constants A', B', C', and F', with the results shown in Table 13.

TABLE 13.—VALUES OF CONSTANTS FOR VARIOUS VALUES OF k.
($g = 62.5$; $w = 150.2$; and $m = 8$)

Constants.	$k = 0.70$.	$k = 0.75$.	$k = 0.80$.
A'	0.22617	0.11718	0.02993
B'	−4.9998	−4.4528	−4.0436
C'	8.5316	7.7501	7.1513
D'	−4.5640	−4.0027	−3.5916

Values of n_y at the up-stream face of the dam were then readily obtained for $k = 0.70$, 0.75, and 0.80. These values, together with those given by Mr. Jakobsen for $k = 0.645$ and 0.844, were plotted against k. This curve gave $k = 0.82$ for $n_y = 0$.

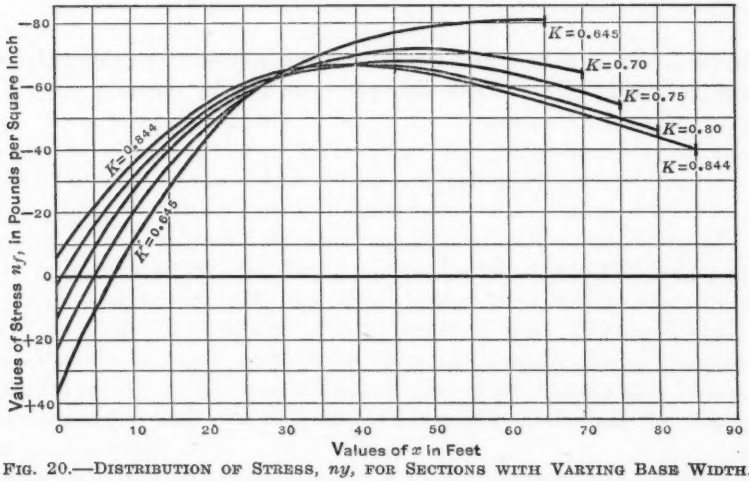

FIG. 20.—DISTRIBUTION OF STRESS, ny, FOR SECTIONS WITH VARYING BASE WIDTH.

The distribution of the n_y-stress was then investigated. The results, combined with those obtained by Mr. Jakobsen for $k = 0.645$ and $k = 0.844$, are shown in Fig. 20. Comparing the n_x-curve for $k = 0.645$ (Fig. 4), with the n_x-curve for $k = 0.844$ (Fig. 7), it is seen that the general shape is very closely the same. This is also true of the corresponding t stress-distribution curves. For this reason the writer has computed values of n_x and t at the toe only. These values, including those given by Mr. Jakobsen for $k = 0.645$ and $k = 0.844$, together with data showing the relationship between n_x n_y and t^2 are shown in Table 14.

The n_y-stresses have been obtained for a dam of triangular section which would be modified in construction by placing additional material at the top. The stress distribution in this modified section could be determined in the following manner, which is based on the fact that Mr. Jakobsen has shown

the stresses due to the weight of the dam alone as computed by the theory of least work to be nearly the same as those computed on the assumption of a linear stress distribution. Compute n_y by the author's method for the triangular section. The stresses due to the added mass of concrete are then computed on the straight-line, stress-distribution theory. The resultant vertical stress will be very closely the algebraic sum of these two.

TABLE 14.—STRESSES AT TOE OF DAM FOR DIFFERENT WIDTHS OF BASE.

($y = 100$ ft.; $b = 100\ k$; $g = 62.5$; $w = 150.2$; $m = 8$; and stress is in pounds per square inch.)

k.	n_y.	n_x.	t.	$n_y\ n_x$.	t^2.
0.645.....	−80.81	−33.62	−52.12	2716.0	2716.0
0.70.....	−65.87	−32.27	−46.10	2125.6	2125.2
0.75.....	−55.16	−31.03	−41.37	1711.6	1711.5
0.80.....	−46.70	−29.89	−37.35	1395.8	1395.0
0.844.....	−40.55	−28.90	−34.24	1171.9	1172.4

Mr. Jakobsen is to be congratulated on having made a valuable addition to present knowledge of the stress distribution in gravity dams.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THEORY OF SIMILARITY AND MODELS

Discussion

BY MESSRS. H. BATEMAN, AND W. P. ROOP

H. BATEMAN,[1] ESQ. (by letter).[2]—A numerical coefficient in a mathematical expression for a property of a complete physical system, S, can depend only on numbers which measure the system relative to itself. These numbers will be called the proportions of the system. In some cases they may be regarded as obtained by means of some measuring apparatus included in the system, but in most cases they are supposed to be known from some former measurements to which the system was subjected and from recognized laws relating to their constancy or mode of variation since these measurements were made.

In an ultimate physical analysis a proportion will be either the ratio of the measures of two entities of the same physical type (for example, two lengths or two velocities), or a definite function of such ratios (a product, for instance). A compound expression such as a Reynolds' number is in this sense a proportion of a system. By using the kinetic theory of gases, Th. von Kármán has shown that the Reynolds number can be regarded as the ratio of two lengths multiplied by the ratio of two velocities. One of the lengths may be taken to be the mean free path of a molecule and one of the velocities the velocity of sound in the fluid.[3]

When a theoretical expression for a numerical coefficient is lacking one cannot strictly speak of a theory of similarity except in particularly simple cases. Generally, one must be content to discuss the validity of the working hypothesis that the coefficient depends essentially upon a certain finite set of proportions which may be called the dominant proportions of the system, S.

A theory of model testing tacitly assumes the existence of motions and configurations which are strictly similar in the geometrical and dynamical senses although sometimes the situation is simplified by the assumption of steady motion and then it may happen that a proportion involving gravity or an

NOTE.—The paper by Benjamin F. Groat, M. Am. Soc. C. E., was published in October, 1930, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussions.

[1] Prof., Guggenheim Graduate School of Aeronautics, California Inst. of Technology, Pasadena, Calif.

[2] Received by the Secretary, November 15, 1930.

[3] Abhandlungen aus dem Aerodynamischen Institut an der Technischen Hochschule, Aachen, Heft 4, p. 25 (1925).

acceleration ceases to be dominant. Much depends, however, upon circumstances.

When a proportion is dominant it is customary to speak of a corresponding effect, such as scale effect, compressibility effect, etc. In this sense one can speak of a gravity or acceleration effect.

The writer agrees with Mr. Groat in his main contentions, but is inclined to say that, although Galileo[4] and Newton recognized the existence of the gravity effect, the credit belongs to W. Froude for the discovery that the associated proportion is often dominant when the physical system involves a solid body and two fluids which are separated by gravity. This discovery was essentially of an experimental nature because generally certain inequalities must be satisfied by the proportions, in order that one may be dominant, or that two may be dominant relative to the others. A valuable discussion of the theoretical basis of model strength tests for aeroplane structures has been given by Messrs. W. L. Cowley and H. Levy.[5] In laying down conditions for the similarity of structures these writers recognize that in many cases there is some latitude in the scope of selection of material for the model.

Physical considerations relating to the structure of matter and the texture of surfaces make it seem improbable that strict similarity can ever be attained in model testing.

When a model has a simple form the number of geometrical proportions needed for its description is small, but in order to specify accurately the shape of an airfoil a large number of proportions are needed. If these proportions are chosen at random without any guiding theory or experience, it is impossible to say which of them will be dominant. A model made with the aid of a template may seem to be of the right shape with a fair degree of accuracy, yet the errors in some proportions needed for the accurate description of the shape, may be large while those in others may be small. If it happens that the great accuracy is attained in the case of unimportant proportions and the low accuracy in the case of proportions which are really dominant, tests with the model will give misleading results.

In hydraulics, model testing is often uncertain on account of the irregular form of the bed of a river and the substance of which it is composed.[6] Some interesting tests of the validity of the theory of similitude in hydraulics have been made by C. Camichel and his co-workers.[7]

W. P. Roop,[8] M. Am. Soc. C. E. (by letter).[8a]—Mr. Groat's effort to make the experimental method based on the theory of similitude more accessible to members of the Society must command the liveliest sympathy from a member of the staff at the United States Experimental Model Basin where the theory of similitude constitutes the "breath of life."

[4] "Growth and Form," by D'Arcy Wentworth Thompson, Cambridge Univ. Press, 1917.

[5] *Engineering*, Vol. 105, p. 219 (1918).

[6] "Hydraulic Laboratory Practice," by John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., Am. Soc. Mech. Eng., 1929.

[7] *Engineering*, Vol. 123, p. 27 (1927); see, also, a paper by C. Camichel, L. Escande, and M. Ricaud, *Revue Générale d'Eléctricité*, t. 20, p. 54 (1926).

[8] Lieut. (C. C.), U. S. N., Washington, D. C.

[8a] Received by the Secretary, December 2, 1930.

There are, however, certain statements in his treatment of the subject which receive a degree of emphasis not altogether appropriate to a dispassionate consideration of the subject. For one, the writer must demur to the author's remark that nothing ever has been added to any of Newton's theories.

In Proposition III Newton[9] establishes the idea of terminal speed of a body falling through a resisting medium, but the writer does not find that he anywhere traces this effect to frictional, as opposed to dynamic, causes. It is not surprising, therefore, that he should believe that the "resistance of globes in infinite mediums is in * * * the duplicate ratio of the velocity * * * "; or that, "a cylinder, a sphere, and a spheroid of equal breadths equally hinder the passage of the water * * *", and that "the forementioned bodies are equally acted on by the water * * *". On the contrary he seems to suggest in the second paragraph of Proposition XL that it is the inertia of the fluid rather than its friction that is primarily concerned.

Detailed discussion of Newton's ideas is perhaps not in order here, but one or two further allusions are necessary. Proposition XL has as its purpose "to find by phenomena the resistance of a globe moving through a perfectly fluid compressed medium". Whether by this he meant an inviscid fluid is not clear. In the discussion of Experiment 12 he uses these words:

"The more swiftly the globes move, the less are they pressed by the fluid at their hinder parts; and if the velocity be perpetually increased, they will at last leave an empty space behind them, unless the compression of the fluid be increased at the same time."

One might suppose from this that Newton was aware of all the intricate processes of eddy formation and cavitation that cause such inconvenient complexity in fluid dynamics.

However, this would be reading into his words, meaning which is certainly not contained in them in an unambiguous way. He did not know that a sphere moving at uniform speed would meet no resistance at all in an infinite frictionless fluid; that its actual resistance, including that due to eddies, owes its origin to the fact that fluids are not frictionless.

Apparently, Mr. Groat still is under the impression that frictional drag on a solid surface due to tangential motion of a fluid across it is ordinarily proportional to the square of the speed. At the Experimental Model Basin such a statement is considered completely erroneous, and if such a law of friction were used, it would lead to gross errors in power estimates.

Possibly some details as to this point may be of interest because they will illustrate the fact that the successful application of the model method depends on development of a suitable experimental technique as surely as upon sound theoretical analysis.

Even Froude,[10] sixty years ago, was aware that the friction on a plank towed edgewise was not proportional to the square of the speed of the plank through the water. He thought that it varied according to a speed exponent which departed from 2.00 by a fixed amount, and by the use of this assumption

[9] "Principia," Book II.

[10] Rept., British Assoc. for the Advancement of Science, 1872 and 1874.

obtained for the first time, in a field where many others had failed, a practicable model method for estimating power requirements for ships.

More recently Gebers,[11] at Vienna, Austria, made careful friction tests and arrived at a result differing somewhat from that of Froude. A large number of friction tests have also been made at the U. S. Experimental Model Basin in the thirty years of its existence.

Data from these three sources have been summarized in a paper recently published,[12] which shows that their precision is not sufficient to permit investigators to distinguish between two types of friction formula, in one of which the exponent may be $\frac{5}{3}$, or even less.

Nevertheless, in spite of arbitrary model scales, which Mr. Groat states are permissible only if the exponent is 2.00, and in spite of variability of actual frictional resistances—in short, in spite of the impossibility of application of the model method for absolute power calculations—it is true that the model method excels all others so long as its function is restricted essentially to that of providing comparisons between existing and proposed full scale projects, not differing too much in type.

Mr. Groat's insistence that dynamic similitude is the one and only kind seems to be somewhat academic. In his "Conclusions" he states: "When that [Newton's] theorem is fully satisfied, Reynolds' number will be satisfied". As Newton says nothing of viscosity, the writer assumes that Mr. Groat must have it in mind that viscous action is itself ultimately a dynamic phenomenon, in which, however, the bodies are of molecular dimensions; but the scale on which molecules are built cannot be controlled. In order that two similar dynamic systems should have at the same time equal speed-length ratios and equal values of Reynolds' number it would be necessary to adjust the kinematic viscosity of the fluid separately for each speed. This is obviously impracticable, and as a matter of fact it is unnecessary. In practical work on ship resistance no attempt is made to satisfy rigorously the conditions of full similitude. It is true that a large part of the uncertainties would be eliminated if rigorous similitude could be achieved; but this cannot be done and engineers are dependent on a friction formula showing variation of friction when equality of Reynolds' numbers is not obtainable.

The really important point is that success in ship-model work is achieved, not by insistence on theoretical rigor, but by acceptance of a procedure that is only partly based on similitude, justified by the fact that it produces the desired results. In all discussions of similitude the fact should not be lost from sight that the power required to drive a ship actually can be predicted with rather high precision by means of model tests. The methods used in accomplishing this are the products of long evolution. It may not be generally known that model tests on ship resistance were made by Benjamin Franklin as early as 1768.

[11] *Schiffbau*, Vol. XXII; translation by National Advisory Committee for Aeronautics, *Technical Memo. 308.*

[12] "Frictional Resistance of Ship Models," by W. P. Roop, M. Am. Soc. C. E., *Transactions*, Soc. of Nav. Archts. and Marine Engrs., Vol. 37 (1929), p. 45.

From the point of view of ship-power estimates the writer would say that the conditions for successful model work include long continuance of model tests under uniform circumstances as to technique, with frequent full-scale comparisons.

A person who expects to be able to draw valid conclusions from one or two or a dozen model tests in a new field in which the possibilities of unforeseen scale effects have not been explored fully had better prepare his mind for disappointment. It is unfortunate that such disappointments in the past have led some engineers to overlook the fact that, under the right conditions, the model method may be made to yield results otherwise unobtainable. Extension of its use on a long-term basis might well lead, as Mr. Groat suggests in his concluding remarks, to the creation of "a new department of engineering design". He is quite right also in referring to the necessity for development of a "corps of specialists". Some progress in this direction is being made at the U. S. Experimental Model Basin and as appreciation of the importance of the subject becomes more general, progress may be expected to become more rapid.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

THE BUTTRESSED DAM OF UNIFORM STRENGTH

Discussion

BY JAMES GIRAND, AND FRED A. NOETZLI

JAMES GIRAND,[1] ASSOC. M. AM. SOC. C. E. (by letter).[1a]—The design of large massive structures has fallen short of the precision developed by the steel designers. There has been much groping for the fundamentals underlying the design of these monolithic masses, in which the weight of the structure is the sole resistance to the load. The author has presented a method, complete in every detail, which appears to be the solution so long sought. Difficulties of design and construction have pointed to a buttress of this shape. Other attempts have been made to approach this method of design, and such a dam is actually under construction on the Choapa Project in Chile in which a graphical method was used instead of the analytic.

In discussion of this paper, two points may be emphasized. The first deals with the factor of safety of the structure, and the second, with the inclined open joints.

Factor of Safety.—It is apparent that in the author's development of the unit column theory and the use of $\gamma_c = 150$ lb. per cu. ft., that each unit column is in equilibrium with the water load balance against its own weight. A decrease of load will allow part of the weight to rest on the underlying columns. An increase in load will overturn the entire column. In other words, the factor of safety of the structure against overturning is 1.0. This is due to the design, considering the full weight of the concrete as effective. Any desired safety factor can be secured by assuming the proper proportion of the concrete to be resisting the load. Various investigations by the writer on the trajectories of stress and the principle of least work have led to the conclusion that at the point of least work the stresses will distribute themselves so that about two-thirds of the weight of the structure will be useful in resisting the load. This gives a logical distribution of stress and a factor of safety of 1.5. This use of two-thirds of the unit weight of the concrete in the design will not materially increase the total quantity of concrete in the buttress; it will change the shape of the catenaries and slightly lengthen

NOTE.—The paper by Herman Schorer, Assoc. M. Am. Soc. C. E., was published in November, 1930, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before other members for further discussions.

[1] Engr., Arizona Hydro-Elec. Syndicate, Phœnix, Ariz.

[1a] Received by the Secretary, December 2, 1930.

the buttress, but it will not increase the thickness. A high factor of safety against overturning is not essential, as the maximum load on a dam is well fixed. If an ample allowance is made for maximum high water, then the only additional load which could occur, would be an earthquake load which is generally accepted at 0.1 *g*.

Inclined Open Joints.—In his paper, the author indicates continuous joints between adjacent column units. This method of putting joints on the lines of first principal stress is being widely accepted, and the Coolidge Dam is an example of buttresses so constructed. This technique offers several inducements to the designer. It simplifies the units of design and permits more accurate use of the older methods. From the construction standpoint, it localizes shrinkage cracks and directs them so as to do the least harm to the structure. The latter is probably the principal excuse for introducing planes of weakness into a structure.

Why should designers intentionally remove some of the inherent strength of the structure? Many dams show considerable tension along the lines of second principal stress, and joints on the lines of first principal stress would seriously weaken the structure. This ability of the concrete to take tension is an additional factor of safety and should not be sacrificed. There are other reasons why the division into columns does not accomplish the best results:

(*a*) If the design is absolutely correct, the first principal stresses uniform, and the second principal stresses zero, it is immaterial whether or not, a joint is present, therefore, it may as well be omitted.

(*b*) The lines of stress will not conform to the shape of the column except at exactly the designed head. If the columns are totally free, this will lead to bending stresses and other undesirable conditions. If the columns are in contact with the adjacent ones, the stresses will be transmitted across the joint, in which case, it would be preferable to have no joint.

(*c*) In any case of a laminated body supporting a load, it is better to put the joints parallel to the lines of second principal stress rather than to the first. This might be exemplified by considering a foot cube of steel necessary to support a load. Such a cube would undoubtedly be made of plates lying flat and not standing on edge.

All these premises lead to the conclusion that no joints should be used; but how should one deal with the concrete as it is being placed to-day? Shrinkage undoubtedly occurs and even the finest structures with exceptionally high test concrete may have high initial tension due to shrinkage. Even if no cracks appear, the tension will radically affect the stresses and make futile the labors of the designer.

As the concrete must be placed in units, these units should be bounded by the lines of first and second principal stress.

The ideal way to place concrete would be in diamond-shaped units, as shown in Fig. 13. As the concrete sets and shrinkage takes place, the entire unit will settle down into the notch. This should relieve all tension in the unit and might actually cause a small initial compression. This method would

not cause as many construction difficulties as would appear at first; the forms could be left open on the top edges until they were half full; strips nailed inside the forms would support short lengths of sheeting put in place as the pour rises. To bound the pour by the lines of first and second principal stress approaches this ideal method and allows the unit poured to free itself of some of the shrinkage stresses. If it is still considered desirable to introduce continuous joints to take up any movements and control cracks, the joints should be introduced along the lines of second principal stress instead of the first. In this location, they will cause the least weakening of the structure.

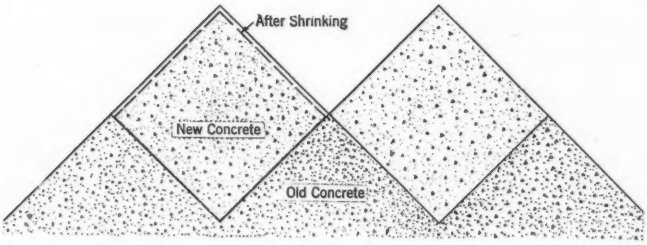


FIG. 13.—AN IDEAL WAY TO PLACE CONCRETE.

The idea of the horizontal construction joint with large form units, which may be sluiced full of concrete, is difficult to dispel. It is a fact, however, that at many dams the bed-rock has been stepped, cleaned, and treated in every manner to assure a good bond and to prevent sliding, while a few feet above it, there is a horizontal construction joint which does not have one-tenth the strength of the bed-rock joint.

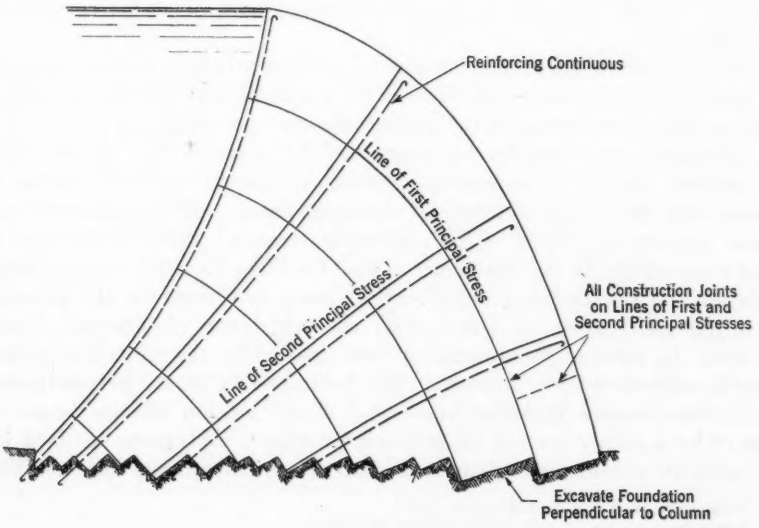

Reinforcing Continuous
Line of First Principal Stress
Line of Second Principal Stress
All Construction Joints on Lines of First and Second Principal Stresses
Excavate Foundation Perpendicular to Column

FIG. 14.

The buttress of absolute minimum weight, as designed by the author, has many advantages. As pointed out, the sliding factor is less than that for a straight-face buttress. The curved up-stream face is also advantageous if a multiple-arch deck is used, as the curve will cause the arches to be of constant intrados. The writer suggests modifications, as shown in Fig. 14. These principally involve: (1) Using a unit weight of about two-thirds that of the concrete for γ_c; (2) the specification that the concrete units of pour shall be bounded by the lines of principal stress; and (3) the use of continuous reinforcement from the bed-rock through to the down-stream face of the buttress.

The author has made a notable and far-reaching contribution to the Engineering Profession.

FRED A. NOETZLI,[2] M. AM. SOC. C. E. (by letter).[2a]—It is a well established fact that the highest degree of economy of material in a structure is obtained when this can be designed so that for full load the material is subjected at all points to the same maximum permissible unit stress. Such ideal conditions are feasible in relatively few engineering structures. The author has shown how they can be approached very closely in dams of the buttressed type.

In the type of dam described by Mr. Schorer a buttress is assumed to consist of a combination of elementary arched column units each of which transmits the portion of the water pressure acting on its up-stream deck, together with its own weight, by uniform axial compression to the foundation. By choosing any desired maximum unit stress of, say, 300, 400, or 500 lb. per sq. in., a designer is enabled with the aid of the author's formulas and diagrams to determine the theoretical dimensions of a buttress of minimum volume for any given height and spacing. Only the tallest buttress at a given dam site must be determined in this manner. The buttresses of lesser height are then merely portions of the tallest buttress cut off at the corresponding elevations.

The theoretical shape of a buttress of uniform strength, such as that shown in Fig. 2, requires some modification for practical applications, for instance, such as that incorporated in the author's design example shown in Fig. 11.

In buttresses of considerable length the shrinkage of the concrete is likely to produce cracks. The author proposes to provide inclined joints, thus eliminating the danger of irregular shrinkage cracks, such as otherwise might occur possibly in planes of high shearing stresses. Such joints have been used successfully in the buttresses of the Coolidge (multiple-dome) Dam, in Arizona, the Big Dalton (multiple-arch) Dam, in California, the Bananeiras (multiple-arch) Dam, in Brazil, and the Rodriguez (Ambursen) Dam, in Mexico. In some of these dams an investigation by the method of principal stresses showed tension stresses in the buttresses in planes perpendicular to the joints (second principal stresses). These tension stresses were taken care of by a proper amount of steel reinforcement. Whenever inclined joints are used in a buttressed dam it is desirable to investigate the stability of

[2] Cons. Hydr. Engr., Los Angeles, Calif.

[2a] Received by the Secretary, December 10, 1930.

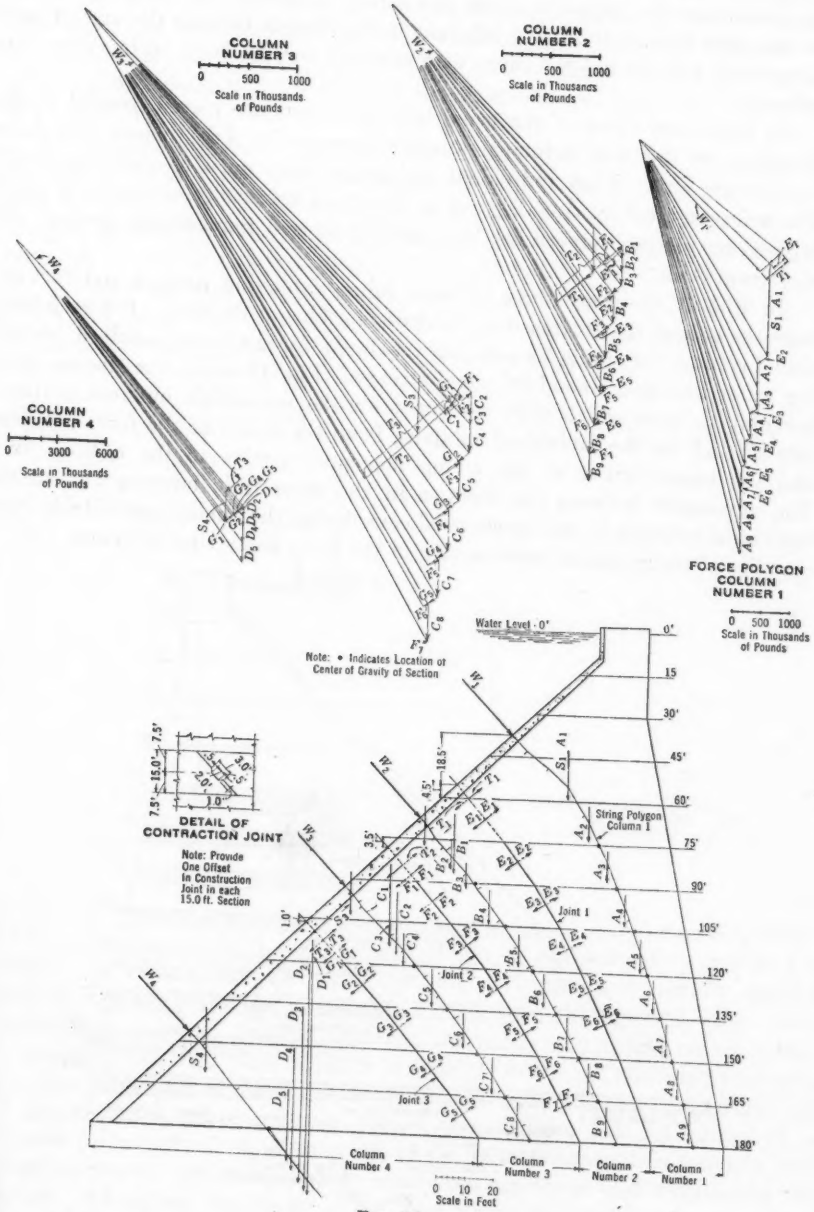

COLUMN NUMBER 3
0 500 1000
Scale in Thousands of Pounds
COLUMN NUMBER 2
0 500 1000
Scale in Thousands of Pounds
COLUMN NUMBER 4
0 3000 6000
Scale in Thousands of Pounds
FORCE POLYGON COLUMN NUMBER 1
0 500 1000
Scale in Thousands of Pounds
Note: • Indicates Location of Center of Gravity of Section
DETAIL OF CONTRACTION JOINT
Note: Provide One Offset in Construction Joint in each 15.0 ft. Section
Water Level - 0'
String Polygon Column 1
Joint 1
Joint 2
Joint 3
Column Number 4
Column Number 3
Column Number 2
Column Number 1
0 10 20
Scale in Feet

FIG. 15.

each individual inclined column. This may be done by combining all the weights and forces, including water pressure, which act upon one column, and then drawing the respective force and string polygons. In the author's type of dam this feature is already inherent in the design, because the axis of each elementary column has the shape of a catenary for the loads acting upon this column.

In buttressed dams of the usual type the joints are placed parallel to the direction of the first principal stresses because in these planes the shear stresses are zero. Each individual inclined column must be stable by itself. The safety against overturning can be increased materially at relatively small expense by placing reinforcing bars parallel to and immediately beneath the up-stream deck.

In Fig. 15 there are three inclined joints extending upward and forward from the base of the buttress to a short distance below the deck. For each individual column the funicular polygon was drawn for the forces involved, including water load and second principal stresses. Fig. 16 shows the stresses upon horizontal planes which were computed for the monolithic buttress (without joints), and for the individual columns, from the forces of the force polygons and the eccentricities of the string polygons relative to the column axes. The agreement between the stresses in the monolithic buttress and in the individual columns is reasonably good considering the almost unavoidable inaccuracies of the graphical construction of the force and string polygons.

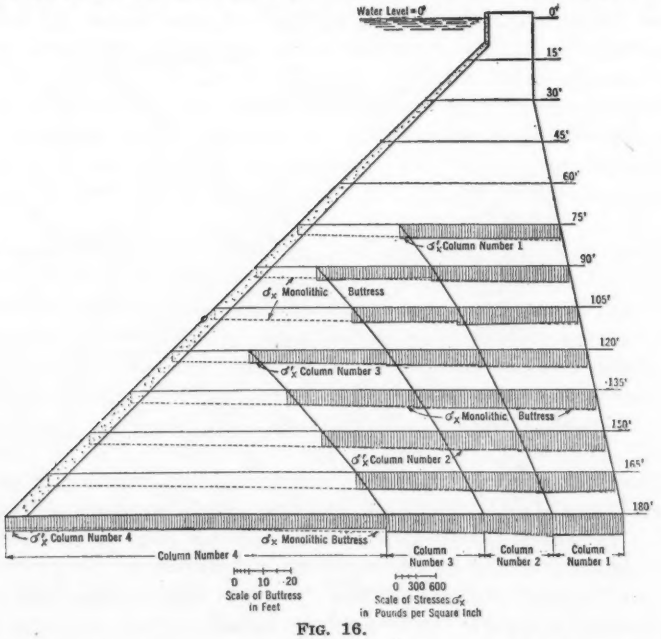


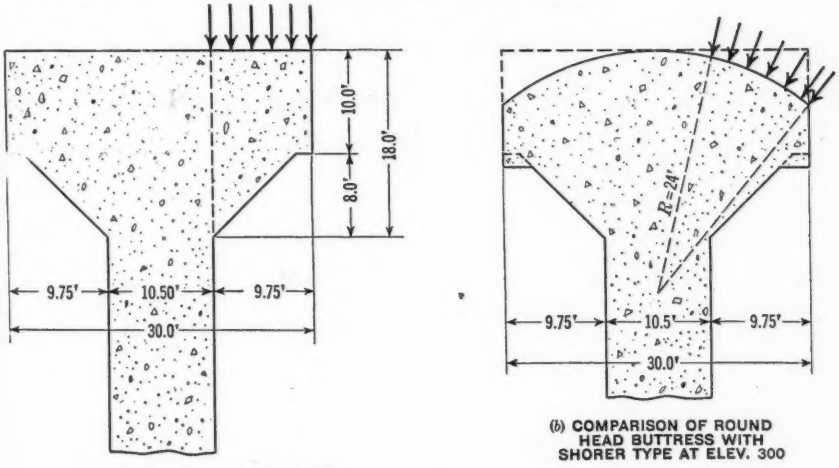
FIG. 16.

The buttress was analyzed further for partial water loads with the reservoir levels 30 ft. and 54.5 ft. below the crest of the dam. Also, for these load

conditions, a reasonably good agreement was found to exist between the stresses in the monolithic buttress and in the individual columns.

In the design example shown in Fig. 11, Mr. Schorer proposes an up-stream deck consisting of a slab cantilevered out laterally for a length of 15 ft. on each side of the center line of the buttress. This cantilever slab is provided with large 45° haunches. The slab, haunches, and buttress wall are constructed integrally.

The normal sections of the deck slab at various elevations are shown by the author in Fig. 11(*h*). For instance, at the 300-ft. depth below the crest the normal section has the dimensions shown in Fig. 17(*a*). The bending moment per foot in slab and haunch on one side of the buttress wall has the value, $M = 62.5 \times 300 \times 9.75^2 \times \frac{1}{2} = 890\,000$ ft-lb. By the ordinary rules of reinforced concrete design the necessary reinforcement would be 3.5 sq. in. per ft. The maximum shear is about 80 lb. per sq. in., requiring bent-up bars or stirrups to ke care of the diagonal tension stresses.



(*a*) SHORER TYPE AT ELEV. 300

(*b*) COMPARISON OF ROUND HEAD BUTTRESS WITH SHORER TYPE AT ELEV. 300

FIG. 17.

By way of comparison there is shown in Fig. 17(*b*) a round-head buttress for the same span of 30 ft. and the same thickness of the buttress used by the author. In such a case the water pressure is transmitted radially and by direct compression through the buttress head upon the buttress wall. Consequently, no bending moment occurs in the buttress head and no steel reinforcement is required. The quantity of concrete in the rounded head compares favorably with that of the cantilever type. Thus, by the simple arrangement of directing the water pressure toward the buttress wall by means of a rounded (or polygonal) up-stream face of the buttress head, plain concrete construction may be used instead of the reinforced cantilever type proposed by the author. At higher elevations of the dam where the water pressure is less, the cantilever type may require less material. On the other hand the principle of the round-head buttress may be applied economically to spans of 50 to 60 ft.,

with a minimum thickness of the buttress walls of 8 to 10 ft. This would enable the dam to be built of mass concrete instead of the thin reinforced concrete usually required for shorter spans.

Of course, the author's ingenious method of determining buttresses of uniform strength containing the minimum possible amount of material for a given design stress, may be used to advantage also for dams of the round-head buttress type.